Journal of Humanitarian, Scientific and Social Sciences
3rd Issue June 2017

DIFFERENTIAL SANDWICH THEOREMS FOR P-VALENT FUNCTIONS DEFINED BY CONVOLUTION INVOLVING CERTAIN FRACTIONAL DERIVATIVE OPERATOR

Somia M. Amsheri
Department of Mathematics, Faculty of Science,
Elmergib University, AL Khums, Libya
somia_amsheri@yahoo.com

Abstract

In the present paper we derive some subordination and superordination results for p-valent functions in the open unit disk defined by convolution involving certain fractional derivative operator. Relevant connections of the results, which are presented in the paper, with various known results are also considered.

Keywords: p-valent function, Hadamard product, incomplete beta function, fractional derivative operators, Differential subordination, Differential superordination,.

Mathematics subject classification: 30C45, 26A33

1. Introduction and Preliminaries

Let $\mathcal{H}(\mathcal{U})$ denote the class of analytic functions in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ and let $\mathcal{H}[a,p]$ denote the subclass of the functions $f \in \mathcal{H}(\mathcal{U})$ of the form:

$$f(z) = a + a_p\, z^p + a_{p+1}\, z^{p+1} + \cdots \qquad (a \in \mathbb{C}, p \in N)$$

Also, let $A(p)$ be the class of functions $f \in \mathcal{H}(\mathcal{U})$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}\, z^{p+n}\ , \qquad p \in N \tag{1.1}$$

and set $A \equiv A(1)$. For functions $f(z) \in A(p)$, given by (1.1), and $g(z)$ given by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n}\, z^{p+n}\ , \qquad p \in N \tag{1.2}$$

The Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} b_{p+n}\, z^{p+n} \qquad z \in \mathcal{U};\ \ p \in N \tag{1.3}$$

Let $f, g \in \mathcal{H}(\mathcal{U})$, we say that the function f is subordinate to g, if there exist a Schwarz function w, analytic in $\mathcal{U}$, with $w(0) = 0$ and $|w(z)| < 1$, $(z \in \mathcal{U})$, such that $f(z) = g(w(z))$ for all $z \in \mathcal{U}$.

This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$. It is well known that, if the function g is univalent in $\mathcal{U}$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Let $p(z), h(z) \in \mathcal{H}(\mathcal{U})$, and let $\Phi(r, s, t;\, z) : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$. If $p(z)$ and $\Phi(p(z), zp'(z),\, z^2\, p''(z);\, z)$ are univalent functions, and if $p(z)$ satisfies the second-order superordination

$$h(z) \prec \Phi(p(z), zp'(z), z^2\, p''(z); z) \tag{1.4}$$

then $p(z)$ is called to be a solution of the differential superordination (1.4). (If $f(z)$ is subordinatnate to $g(z)$, then $g(z)$ is called to be superordinate to $f(z)$). An analytic function $q(z)$ is called a subordinant if $q(z) \prec p(z)$ for all $p(z)$ satisfies (1.4) . An univalent subordinant $\tilde{q}(z)$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinants $q(z)$ of (1.4) is said to be the best subordinant.

Recently, Miller and Mocanu [7] obtained conditions on $h(z), q(z)$ and Φ for which the following implication holds true:

$$h(z) \prec \Phi(p(z), zp'(z), z^2\, p''(z); z) \Longrightarrow q(z) \prec p(z)$$

with the results of Miller and Mocanu [7], Bulboaca [3] investigated certain classes of first

order differential superordinations as well as superordination-preserving integral operators

[4]. Ali et al. [2] used the results obtained by Bulboaca [4] and gave the sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in $\mathcal{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$. Shanmugam et al. [13] obtained sufficient conditions for a normalized analytic functions to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{(f(z))^2} \prec q_2(z)$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in $\mathcal{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$.

Let ${}_2F_1(a, b; c; z)$ be the Gauss hypergeometric function defined for $z \in \mathcal{U}$ by, (see Srivastava and Karlsson [17])

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n\; z^n}{(c)_n\; n!} \tag{1.5}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & , \quad n = 0 \\ \lambda(\lambda+1)(\lambda+2)\ldots.(\lambda+n-1) & , \quad n \in \mathbb{N} \end{cases} \tag{1.6}$$

for $\lambda \neq 0, -1, -2, \ldots$

We recall the following definitions of fractional derivative operators which were used

by Owa [10], (see also [11]) as follows:

Definition 1.1 The fractional derivative operator of order λ is defined by,

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}}\, d\xi \tag{1.7}$$

where $0 \leq \lambda < 1$, $f(z)$ is analytic function in a simply- connected region of the z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 1.2 Let $0 \leq \lambda < 1$, and $\mu, \eta \in \mathbb{R}$. Then, in terms of the familiar Gauss's hypergeometric function ${}_2F_1$, the generalized fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}$ is

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{d}{dz}\left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)}\int_0^z (z-\xi)^{-\lambda} f(\xi)\, {}_2F_1\left(\mu-\lambda, 1-\eta; 1-\lambda; 1-\frac{\xi}{z}\right) d\xi\right) \quad (1.8)$$

where $f(z)$ is analytic function in a simply- connected region of the z-plane containing the origin with the order $f(z) = O(|z|^{\varepsilon})$, $z \to 0$, where $\varepsilon > \max\{0, \mu-\eta\} - 1$, and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 1.3 Under the hypotheses of Definition 1.2, the fractional derivative operator $J_{0,z}^{\lambda+m,\mu+m,\eta+m} f(z)$ of a function $f(z)$ is defined by

$$J_{0,z}^{\lambda+m,\mu+m,\eta+m} f(z) = \frac{d^m}{dz^m} J_{0,z}^{\lambda,\mu,\eta} f(z) \quad (1.9)$$

Notice that

$$J_{0,z}^{\lambda,\lambda,\eta} f(z) = D_z^{\lambda} f(z), \qquad 0 \leq \lambda < 1 \quad (1.10)$$

With the aid of the above definitions, we define a modification of the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$ by

$$M_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\eta)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)} z^{\mu} J_{0,z}^{\lambda,\mu,\eta} f(z) \tag{1.11}$$

for $f(z) \in A(p)$ and $\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1; p \in N$. Then it is observed that $M_{0,z}^{\lambda,\mu,\eta} f(z)$ maps $A(p)$ onto itself as follows:

$$M_{0,z}^{\lambda,\mu,\eta} f(z) = z^p + \sum_{n=1}^{\infty} \frac{(p+1)_n (p+1-\mu+\eta)_n}{(p+1-\mu)_n (p+1-\lambda+\eta)_n} a_{p+n} z^{p+n} \tag{1.12}$$

Let $\varphi_p(a,c;z)$ be the incomplete beta function defined for $z \in \mathcal{U}$ by

$$\varphi_p(a,c;z) = z^p + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{p+n} \quad ,p \in N \tag{1.13}$$

where $a \in \mathbb{R}$, $c \in \mathbb{R}\backslash\{0,-1,-2,\dots\}$ and $(\lambda)_n$ is given by (1.6), and using the Hadamard product, we define the following operator $\Omega_p^{\lambda,\mu,\eta} f(z): \mathcal{U} \to \mathcal{U}$ by

$$\Omega_p^{\lambda,\mu,\eta} f(z) = \varphi_p(a,c;z) * M_{0,z}^{\lambda,\mu,\eta} f(z) \tag{1.14}$$

If $f(z) \in A(p)$, then from (1.12) and (1.14), we can easily see that

$$\Omega_p^{\lambda,\mu,\eta} f(z) = z^p + \sum_{n=1}^{\infty} \frac{(a)_n (p+1)_n (p+1-\mu+\eta)_n}{(c)_n (p+1-\mu)_n (p+1-\lambda+\eta)_n} a_{p+n} z^{p+n} \tag{1.15}$$

where $a \in \mathbb{R}$, $c \in \mathbb{R}\backslash\{0,-1,-2,\dots\}$, $\lambda \geq 0$, $\mu < p+1$, $\eta > \max(\lambda,\mu) - p - 1$ and $p \in N$.

Notice that, if $a = c$, then we have $\Omega_p^{\lambda,\mu,\eta} f(z) = M_{0,z}^{\lambda,\mu,\eta} f(z)$, $\Omega_p^{0,0,\eta} f(z) = f(z)$ and $\Omega_p^{1,1,\eta} f(z) = \frac{zf'(z)}{p}$. Also, we observe that the operator $\Omega_p^{\lambda,\mu,\eta} f(z)$ reduces to the following operators considered earlier for different choices of λ, μ, a and c .

(1) For $\lambda = \mu = 0$, we get the operator $L_p(a,c)f(z)$ which is motivated from Carlson-Shaffer operator [5].

(2) For $\lambda = \mu = 0$, $a = m + p$ and $c = 1$, we get the operator D^{m+p-1} which is the Ruscheweyh derivative operator of order $m + p - 1$ (see [12, 15]).

It is easily verified from (1.15) that

$$z\left(\Omega_p^{\lambda,\mu,\eta} f(z)\right)' = (p-\mu)\, \Omega_p^{\lambda+1,\mu+1,\eta+1} f(z) + \mu\, \Omega_p^{\lambda,\mu,\eta} f(z) \qquad (1.16)$$

The object of this paper is to derive several subordination and superordination results defined by Hadamard product involving certain fractional derivative operator. Furthermore, we obtain the previous results of Aouf et al. [1], Shammugam et al. [14], Obradovic et al. [8], Obradovic and Owa [9], and Srivastava and Lashin [16] as special cases of some the results presented here.

In order to prove our results we mention to the following known results which shall be used in the sequel.

Lemma 1.4 [11] *Let* $\lambda, \mu, \eta \in \mathbb{R}$, *such that* $\lambda \geq 0$ *and* $k > max\{0, \mu - \eta\} - 1$. *Then*

$$J_{0,z}^{\lambda,\mu,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\mu+\eta+1)}{\Gamma(k-\mu+1)\Gamma(k-\lambda+\eta+1)} z^{k-\mu} \tag{1.17}$$

Definition 1.5 [7] *Denoted by* Q *the set of all functions* f *that are analytic and injective in* $\bar{\mathcal{U}} - E(f)$ *where*

$$E(f) = \left\{\xi \in \partial\mathcal{U} : \lim_{z\to\xi} f(z) = \infty\right\}$$

and are such that $f'(\xi) \neq 0$ *for* $\xi \in \partial\mathcal{U} - E(f)$.

Lemma 1.6 [6] *Let the function* q *be univalent in the open unit disk* $\mathcal{U}$, *and* θ *and* φ *be analytic in a domain* D *containing* $q(\mathcal{U})$ *with* $\varphi(w) \neq 0$ *when* $w \in q(\mathcal{U})$. *Set* $Q(z) = zq'(z)\varphi(q(z))$ *and* $h(z) = \theta(q(z)) + Q(z)$. *Suppose that*

1. Q *is starlike univalent in* $\mathcal{U}$, *and*
2. $Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ *for* $z \in \mathcal{U}$

If

$$\theta\big(p(z)\big) + zp'(z)\varphi(p(z)) \prec \theta\big(q(z)\big) + zq'(z)\varphi(q(z))$$

Then $p(z) \prec q(z)$ *and* q *is the best dominant.*

Lemma 1.7 [3] *Let the function* q *be univalent in the open unit disk* $\mathcal{U}$, *and* θ *and* φ *be analytic in a domain* D *containing* $q(\mathcal{U})$ *with* $\varphi(w) \neq 0$ *when* $w \in q(\mathcal{U})$. *Suppose that*

1. $Re\left(\frac{\theta'(q(z))}{\varphi(q(z))}\right) > 0$ *for* $z \in \mathcal{U}$
2. $zq'(z)\varphi\big(q(z)\big)$ *is starlike univalent in* $\mathcal{U}$.

If $p(z) \in \mathcal{H}[q(0),1] \cap Q$ *with* $p(\mathcal{U}) \subseteq D$, *and* $\theta\big(p(z)\big) + zp'(z)\varphi(p(z))$ *is univalent in* $\mathcal{U}$, *and*

$$\theta\big(q(z)\big) + zq'(z)\varphi(q(z)) \prec \theta\big(p(z)\big) + zp'(z)\varphi(p(z))$$

then $q(z) \prec p(z)$ *and* q *is the best subordinant.*

2. Subordination and superordination for p-valent functions

We begin with the following result involving differential subordination between analytic functions.

Theorem 2.1. *Let* $\left(\frac{\Omega_p^{\lambda,\mu,\eta} f(z)}{z^p}\right)^{\gamma} \in \mathcal{H}(\mathcal{U})$ *and let the function* $q(z)$ *be analytic and univalent in* $\mathcal{U}$ *such that* $q(z) \neq 0, (z \in \mathcal{U})$. *Suppose that* $\frac{zq'(z)}{q(z)}$ *is starlike and univalent in* $\mathcal{U}$. *Let*

$$Re\left\{1 + \frac{\xi}{\beta} q(z) + \frac{2\delta}{\beta}\big(q(z)\big)^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0 \qquad (2.1)$$

$$(\alpha, \delta, \xi, \beta \in \mathbb{C};\ \beta \neq 0)$$

and $f(z) \in A(p)$, *and*

$$\Psi_{\lambda,\mu,\eta}(\gamma,\xi,\beta,\delta,f)(z) = \alpha + \xi\left(\frac{\Omega_p^{\lambda,\mu,\eta} f(z)}{z^p}\right)^{\gamma} + \delta\left(\frac{\Omega_p^{\lambda,\mu,\eta} f(z)}{z^p}\right)^{2\gamma}$$

$$+\beta\gamma(p-\mu)\left[\frac{\Omega_p^{\lambda+1,\mu+1,\eta+1} f(z)}{\Omega_p^{\lambda,\mu,\eta} f(z)} - 1\right] \qquad (2.2)$$

If q *satisfies the following subordination:*

$$\Psi_{\lambda,\mu,\eta}(\gamma,\xi,\beta,\delta,f)(z) \prec \alpha + \xi q(z) + \delta\big(q(z)\big)^2 + \beta\frac{zq'(z)}{q(z)}$$

$(\lambda \geq 0;\ \ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in N;\ \alpha,\delta,\xi,\gamma,\beta \in \mathbb{C};\ \gamma \neq 0;\ \beta \neq 0)$

then

$$\left(\frac{\Omega_p^{\lambda,\mu,\eta} f(z)}{z^p}\right)^{\gamma} \prec q(z) \qquad (\gamma \in \mathbb{C}\backslash\{0\}) \tag{2.3}$$

and q is the best dominant.

Proof. Let the function $p(z)$ be defined by

$$p(z) = \left(\frac{\Omega_p^{\lambda,\mu,\eta} f(z)}{z^p}\right)^{\gamma} \ (z \in \mathcal{U}\backslash\{0\};\ \ \gamma \in \mathbb{C}\backslash\{0\})$$

So that, by a straightforward computation, we have

$$\frac{zp'(z)}{p(z)} = \gamma\left[\frac{z\left(\Omega_p^{\lambda,\mu,\eta} f(z)\right)'}{\Omega_p^{\lambda,\mu,\eta} f(z)} - p\right]$$

By using the identity (1.16) we obtain

$$\frac{zp'(z)}{p(z)} = \gamma\left[(p-\mu)\frac{\Omega_p^{\lambda+1,\mu+1,\eta+1} f(z)}{\Omega_p^{\lambda,\mu,\eta} f(z)} - (p-\mu)\right]$$

By setting $\theta(w) = \alpha + \xi w + \delta w^2$ and $\varphi(w) = \frac{\beta}{w}$, it can be easily verified that θ is analytic in $\mathbb{C}$, φ is analytic in $\mathbb{C} \setminus \{0\}$, and that $\varphi(w) \neq 0,\ (w \in \mathbb{C}\setminus\{0\})$. Also, by letting

$$Q(z) = zq'(z)\varphi\big(q(z)\big) = \beta\frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta\big(q(z)\big) + Q(z) = \alpha + \xi q(z) + \delta\big(q(z)\big)^2 + \beta\frac{zq'(z)}{q(z)}$$

we find that $Q(z)$ is starlike univalent in $\mathcal{U}$ and that

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{1 + \frac{\xi}{\beta}q(z) + \frac{2\delta}{\beta}\big(q(z)\big)^2 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0$$

The assertion (2.3) of Theorem 2.1 now follows by an application of Lemma 1.6.

Remark 1. For the choices $q(z) = \frac{1+Az}{1+\mathrm{Bz}}$, $-1 \leq B < A \leq 1$ and $q(z) = \left(\frac{1+z}{1-\mathrm{z}}\right)^{\varepsilon}$, $0 < \varepsilon \leq 1$, in Theorem 2.1, we get the following results (Corollary 2.2 and Corollary 2.3) below.

Corollary 2.2. *Assume that* (2.1) *holds. If* $f \in A(p)$, *and*

$$\Psi_{\lambda,\mu,\eta}(\gamma,\xi,\beta,\delta,f)(z) \prec \alpha + \xi\left(\frac{1+Az}{1+\mathrm{Bz}}\right) + \delta\left(\frac{1+Az}{1+\mathrm{Bz}}\right)^2 + \frac{\beta(A-B)z}{(1+Az)(1+Bz)}$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in N;\ \alpha,\delta,\xi,\gamma,\beta \in \mathbb{C};\gamma \neq 0;\ \beta \neq 0)$$

where $\Psi_{\lambda,\mu,\eta}(\gamma,\xi,\beta,\delta,f)(z)$ *is as defined in* (2.2), *then*

$$\left(\frac{\Omega_p^{\lambda,\mu,\eta}f(z)}{z^p}\right)^{\gamma} \prec \frac{1+Az}{1+\mathrm{Bz}} \qquad (\gamma \in \mathbb{C}\backslash\{0\})$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Corollary 2.3. *Assume that* (2.1) *holds. If* $f \in A(p)$, *and*

$$\Psi_{\lambda,\mu,\eta}(\gamma,\xi,\beta,\delta,f)(z) \prec \alpha + \xi\left(\frac{1+z}{1-\mathrm{z}}\right)^{\varepsilon} + \delta\left(\frac{1+z}{1-\mathrm{z}}\right)^{2\varepsilon} + \frac{2\beta\varepsilon z}{1-z^2}$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in N;\ \alpha,\delta,\xi,\gamma,\beta \in \mathbb{C};\gamma \neq 0;\ \beta \neq 0)$$

where $\Psi_{\lambda,\mu,\eta}(\gamma,\xi,\beta,\delta,f)(z)$ *is as defined in* (2.2), *then*

$$\left(\frac{\Omega_p^{\lambda,\mu,\eta} f(z)}{z^p}\right)^{\gamma} \prec \left(\frac{1+z}{1-\mathrm{z}}\right)^{\varepsilon} \qquad (\gamma \in \mathbb{C}\backslash\{0\})$$

and $\left(\frac{1+z}{1-\mathrm{z}}\right)^{\varepsilon}$ *is the best dominant.*

Remark 2. Taking $p = 1, a = b + 1$ and $\lambda = \mu = 0$ in Theorem 2.1, Corollary 2.2 and Corollary 2.3, we obtain the subordination results for linear operator due to Shammngam et al. [[14], Theorem 3, Corollary 1 and Corollary 2, respectively].

Remark 3. For the choice $q(z) = \frac{1}{(1-\mathrm{z})^{2\mathrm{AB}}}$ $(A, B \in \mathbb{C}\backslash\{0\})$; $a = c,\ \lambda = \mu = \delta = \xi = 0;\ p = \alpha = 1;\ \gamma = A$ and $\beta = \frac{1}{AB}$ in Theorem 2.1, we obtain the following known result due to Obradovic et al. [8].

Corollary 2.4. *Let* $A, B \in \mathbb{C}\backslash\{0\}$ *such that* $|2AB - 1| \leq 1$ *or* $|2AB + 1| \leq 1$ *If* $f \in A$, *and*

$$1+\frac{1}{B}\left(\frac{zf'(z)}{f(z)}-1\right)\prec\frac{1+z}{1-z}$$

then

$$\left(\frac{f(z)}{z}\right)^{A}\prec\frac{1}{(1-z)^{2AB}}$$

and $\frac{1}{(1-z)^{2AB}}$ *is the best dominant.*

Remark 4. For $A=1$, Corollary 2.4 reduces to the recent result of Srivastava and Lashin [16].

Remark 5. For the choice $q(z)=(1+Bz)^{\frac{\gamma(A-B)}{B}}$; $a=c,\ \lambda=\mu=\delta=\xi=0$; $p=\alpha=1$ and $\beta=1$ in Theorem 2.1, we obtain the following known result due to Obradovic and Owa [9].

Corollary 2.5. *Let* $-1\leq A<B\leq 1$ *with* $B\neq 0$ *and suppose that* $\left|\frac{\gamma(A-B)}{B}-1\right|\leq 1$ *or* $\left|\frac{\gamma(A-B)}{B}+1\right|\leq 1,\ \gamma\in\mathbb{C}\backslash\{0\}$. *If* $f\in A$, *and*

$$1+\gamma\left(\frac{zf'(z)}{f(z)}-1\right)\prec\frac{1+[B+\gamma(A-B)]z}{1+Bz}$$

then

$$\left(\frac{f(z)}{z}\right)^{\gamma}\prec(1+Bz)^{\frac{\gamma(A-B)}{B}}$$

and $(1+Bz)^{\frac{\gamma(A-B)}{B}}$ *is the best dominant.*

Remark 6. For the choice $q(z)=\frac{1}{(1-z)^{2ABe^{-i\tau}\cos\tau}}$ $(A,B\in\mathbb{C}\backslash\{0\});|\tau|<\frac{\pi}{2};$ $a=c;$ $\lambda=\mu=\delta=\xi=0;$ $p=\alpha=1;$ $\gamma=A$ and $\beta=\frac{e^{i\tau}}{AB\cos\tau}$ in Theorem 2.1, we get the following result due to Aouf et al. [1].

Corollary 2.5. *Let* $A,B\in\mathbb{C}\backslash\{0\}$ *and* $|\tau|<\frac{\pi}{2}$ *, and suppose that* $\left|2ABe^{-i\tau}\cos\tau-1\right|\leq 1$ *or* $\left|2ABe^{-i\tau}\cos\tau+1\right|\leq 1$. *Let* $f\in A$, *and suppose that* $\frac{f(z)}{z}\neq 0$ *for all* $z\in\mathcal{U}$. *If*

$$1+\frac{e^{i\tau}}{B\cos\tau}\left(\frac{zf'(z)}{f(z)}-1\right)\prec\frac{1+z}{1-z}$$

then

$$\left(\frac{f(z)}{z}\right)^{A}\prec\frac{1}{(1-z)^{2ABe^{-i\tau}\cos\tau}}$$

and $\frac{1}{(1-z)^{2ABe^{-i\tau}\cos\tau}}$ *is the best dominant.*

Next, by appealing to Lemma 1.7 of the preceding section, we prove the following.

Theorem 2.7. *Let* q *be analytic and univalent in* $\mathcal{U}$ *such that* $q(z)\neq 0$ *and* $\frac{zq'(z)}{q(z)}$ *be starlike and univalent in* $\mathcal{U}$. *Further, let us assume that*

$$Re\left\{\frac{2\delta}{\beta}\big(q(z)\big)^{2}+\frac{\xi}{\beta}q(z)\right\}>0\,,\qquad(\delta,\xi,\beta\in\mathbb{C};\ \beta\neq 0)\qquad(2.4)$$

If $f(z)\in A(p)$,

$$0 \neq \left(\frac{\Omega_p^{\lambda,\mu,\eta} f(z)}{z^p}\right)^{\gamma} \in \mathcal{H}[q(0),1] \cap Q$$

and $\Psi_{\lambda,\mu,\eta}(\gamma,\xi,\beta,\delta,f)(z)$ is *univalent in* $\mathcal{U}$, *then*

$$\alpha + \xi q(z) + \delta\big(q(z)\big)^2 + \beta\frac{zq'(z)}{q(z)} \prec \Psi_{\lambda,\mu,\eta}(\gamma,\xi,\beta,\delta,f)(z)$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in N;\ \alpha,\delta,\xi,\gamma,\beta \in \mathbb{C}; \gamma \neq 0;\ \beta \neq 0)$$

implies

$$q(z) \prec \left(\frac{\Omega_p^{\lambda,\mu,\eta} f(z)}{z^p}\right)^{\gamma} \qquad (\gamma \in \mathbb{C}\backslash\{0\}) \tag{2.5}$$

and q *is the best subordinant where* $\Psi_{\lambda,\mu,\eta}(\gamma,\xi,\beta,\delta,f)(z)$ *is as defined in* (2.2).

Proof. By setting $\theta(w) = \alpha + \xi w + \delta w^2$ and $\varphi(w) = \frac{\beta}{w}$, it can be easily observed that θ is analytic in $\mathbb{C}$, φ is analytic in $\mathbb{C} \setminus \{0\}$, and that $\varphi(w) \neq 0$, $(w \in \mathbb{C}\setminus \{0\})$. Since q is convex (univalent) function it follows that,

$$Re\left\{\frac{\theta'(q(z))}{\varphi(q(z))}\right\} = Re\left\{\frac{2\delta}{\beta}\big(q(z)\big)^2 + \frac{\xi}{\beta}q(z)\right\} > 0$$

$$(\delta,\xi,\beta \in \mathbb{C};\ \beta \neq 0)$$

The assertion (2.5) of Theorem 2.7 now follows by an application of Lemma 1.7.

Combining Theorem 2.1 and Theorem 2.7, we get the following sandwich theorem.

Theorem 2.8. *Let q_1 and q_2 be univalent in $\mathcal{U}$ such that $q_1(z) \neq 0$ and $q_2(z) \neq 0,\ z \in \mathcal{U}$ with $\frac{zq_1'(z)}{q_1(z)}$ and $\frac{zq_2'(z)}{q_2(z)}$ being starlike univalent. Suppose that q_1 satisfies* (2.4) *and q_2 satisfies* (2.1). *If $f(z) \in A(p)$,*

$$\left(\frac{\Omega_p^{\lambda,\mu,\eta} f(z)}{z^p}\right)^{\gamma} \in \mathcal{H}[q(0),1] \cap Q$$

and $\Psi_{\lambda,\mu,\eta}(\gamma,\xi,\beta,\delta,f)(z)$ is *univalent in $\mathcal{U}$, then*

$$\alpha + \xi q_1(z) + \delta\big(q_1(z)\big)^2 + \beta\frac{zq_1'(z)}{q_1(z)} \prec \Psi_{\lambda,\mu,\eta}(\gamma,\xi,\beta,\delta,f)(z)$$

$$\prec \alpha + \xi q_2(z) + \delta\big(q_2(z)\big)^2 + \beta\frac{zq_2'(z)}{q_2(z)}$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\mu) - p - 1;\ p \in N;\ \alpha,\delta,\xi,\gamma,\beta \in \mathbb{C};\gamma \neq 0;\ \beta \neq 0)$$

implies

$$q_1(z) \prec \left(\frac{\Omega_p^{\lambda,\mu,\eta} f(z)}{z^p}\right)^{\gamma} \prec q_2(z) \qquad (\gamma \in \mathbb{C}\backslash\{0\}) \qquad (2.6)$$

and q_1 and q_2 are respectively the best subordinant and the best dominant where $\Psi_{\lambda,\mu,\eta}(\gamma,\xi,\beta,\delta,f)(z)$ is as defined in (2.2).

Remark 7. Taking $p = 1, a = b + 1$ and $\lambda = \mu = 0$ in Theorem 2.6 and Theorem 2.7, we obtain the superordination and sandwich results for linear operator due to Shammngam et al. [[14], Theorem 4 and Theorem 5, respectively].

Remark 8. For $\lambda = \mu = 0$ and $a = c$ in Theorem 2.8, we get the following result.

Theorem 2.9. *Let q_1 and q_2 be univalent in $\mathcal{U}$ such that $q_1(z) \neq 0$ and $q_2(z) \neq 0$, $z \in \mathcal{U}$ with $\frac{zq_1'(z)}{q_1(z)}$ and $\frac{zq_2'(z)}{q_2(z)}$ being starlike univalent. Suppose that q_1 satisfies* (2.4) *and q_2 satisfies* (2.1). *If $f(z) \in A(p)$,*

$$\left(\frac{f(z)}{z^p}\right)^{\gamma} \in \mathcal{H}[q(0), 1] \cap Q$$

and let

$$\Psi_1(\gamma, \xi, \beta, \delta, f)(z) = \alpha + \xi\left(\frac{f(z)}{z^p}\right)^{\gamma} + \delta\left(\frac{f(z)}{z^p}\right)^{2\gamma} + \beta\gamma\left(\frac{zf'(z)}{f(z)} - p\right)$$

is *univalent in $\mathcal{U}$, then*

$$\alpha + \xi q_1(z) + \delta\big(q_1(z)\big)^2 + \beta\frac{zq_1'(z)}{q_1(z)} \prec \Psi_1(\gamma, \xi, \beta, \delta, f)(z)$$

$$\prec \alpha + \xi q_2(z) + \delta\big(q_2(z)\big)^2 + \beta\frac{zq_2'(z)}{q_2(z)}$$

$$(p \in N;\ \alpha, \delta, \xi, \gamma, \beta \in \mathbb{C};\ \gamma \neq 0;\ \beta \neq 0)$$

implies

$$q_1(z) \prec \left(\frac{f(z)}{z^p}\right)^{\gamma} \prec q_2(z) \qquad (\gamma \in \mathbb{C}\backslash\{0\}) \tag{2.7}$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

References

[1] Aouf, M. K., AL-Oboudi, F. M., and Haidan, M. M., 2005, "On Some Results for λ -Apiralike and λ-Robertson Functions of Complex Order", Publ. Inst. Math. (Belgrade), 77(91), pp. 93-98.

[2] Ali, R. M., Ravichandran, V., Hussain, K. M., and Subramaniant, K. G., 2005, "Differential Sandwich Theorems for Certain Analytic Functions", Far East J. Math. Sci. 15(1), pp. 87-94.

[3] Bulboaca, T. A., 2002, "Classes of First-Order Differential Superordinations", Demonstr. Math. 35(2), pp. 287-292.

[4] Bulboaca, T. A., 2002, "A Class of Superordination-Preserving Integral Operators", Indag. Math. New Ser. 13(3), pp. 301-311.

[5] Carlson, B. C., and Shaffer, D. B., 1984, "Starlike and Prestarlike Hypergeometric Functions", SIAM, J. Math. Anal, 15(4), pp. 737-745.

[6] Miller, S. S., and Bulboaca, T. A., 2000, "Differential Subordinations: Theory and Applications", Pure and Applied Mathematics, No. 225, Marcel Dekker, New York.

[7] Miller, S. S., and Mocanu, B. T., 2003, "Subordinations of Differential Superordinations", Complex Variables, 48(10), pp. 815-826.

[8] Obradovic, M., Aouf, M. K., and Owa, S., 1989, "On Some Results for Starlike Functions of Complex Order", Publ. Inst. Math. (Beograd) (N. S.), 46(60), PP. 79-85.

[9] Obradovic, M., and Owa, S., 1990, "On Certain Properties for Some Classes of Starlike Functions", J. Math. Anal. Appl., 145, pp. 357-364.

[10] Owa, S., 1978, “On The Distortion Theorems- I”, Kyungpook. Math. J. 18, pp. 53-59.

[11] Raina, R. K., and Srivastava, H. M., 1996, “A Certain Subclass of Analytic Functions Associated With Operators of Fractional Calculus”, Computers & Mathematics with Applications, 32, pp. 13-19.

[12] Ruscheweyh, S., 1975, “New Crieteria for Univalent Functions”, Proc. Amer. Math. Soc. 49, pp. 109-115.

[13] Shanmugam, T. N., Ravichandran, V., and Sivasubramanian, S., 2006, “Differential Sandwich Theorems for Some Subclasses of Analytic Functions”, Austral J. Math. Anal. Appl. 3(1), pp. 1-11.

[14] Shanmugam, T. N., Ramachandran, G., Darus, M., and Sivasubramanian, S., 2007, “Differential Sandwich Theorems for Some Subclasses of Analytic Functions Involving a Linear Operator”, Acta. Math. Univ. Comenianae. Vol. LXXVI, (2), pp. 287-294.

[15] Srivastava, H. M., and Owa, S., Editors, 1992, “Current Topics in Analytic Function Theory”, World Scientific, Singapore.

[16] Srivastava, H. M., and Lashin, A. Y., 2005, “Some Applications of the Briot-Bouquet Differential Subordination”, J. Inequal. Pure Appl. Math. 6(2), Article 41, 7pp.

[17] Srivastava, H. M., and Karlsson, P. M., 1985, “Multiple Gaussian Hypergeometric Series”, Halsted Press (Ellis Horwood Limited, Chichester), Wiley, New York/ Chichester/ Brishane/ Toronto.