A COMPARISON BETWEEN ELZAKI TRANSFORM AND LAPLACE TRANSFORM TO SOLVE ODE'S AND SYSTEMS OF ODE'S

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Keywords: Laplace transform , Elzaki transform, Differential Equation

Abstract: In this paper Elzaki transform was compared with Laplace transform in solving some ODE's and systems of ODE's

1.Introduction

The term "differential equations" was proposed in 1676 by G. Leibniz. The first studies of these equations were carried out in the late 17th century in the context of certain problems in mechanics and geometry. Ordinary differential equations have important applications and are a powerful tool in the study of many problems in the natural sciences and in technology; they are extensively employed in mechanics, astronomy, physics, and in many problems of chemistry and biology. The reason for this is the fact that objective laws governing certain phenomena can be written as ordinary differential equations, so that the equations themselves are a quantitative expression of these laws. For instance, Newton's laws of mechanics make it possible to reduce the description of the motion of mass points or solid bodies to solving ordinary differential equations. The computation of radio technical circuits or satellite trajectories, studies of the stability of a plane in flight, and explaining the course of chemical reactions are all carried out by studying and solving ordinary differential equations. The most interesting and most important applications of these equations are in the theory of oscillations and in automatic control theory. Applied problems in turn produce new formulations of problems in the theory of ordinary differential equations; the mathematical theory of optimal control in fact arose in this manner.

Integral transform method is widely used to solve the several differential equations with the initial values or boundary conditions. In the literature there are numerous integral transforms such as the Laplace, Fourier, Mellin, and Hankel , etc.

Elzaki transform which is a modified general Laplace and Sumudu transforms, has been shown to solve effectively, easily and accurately a large class of Linear Differential Equations. Elzaki Transform was successfully applied to integral equations, partial differential equations, ordinary differential equations with variable coefficients and system of all these equations. The purpose of this paper is to solve Ordinary Differential Equations (ODE's) and systems of Ordinary Differential Equations with Laplace transform and Elzaki transform to make a comparison between them.

2. Definitions and Standard Results

2.1 The Laplac.e Transform

Definition: If $f(t)$ is a function defined for all positive values of t, then the Laplace transform of it is defined as

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)dt \qquad (1)$$

provided that the integral exists. Here the parameter s is a real or complex number. The corresponding inverse Laplace transform is $\mathcal{L}^{-1}[F(s)] = f(t)$. Here $f(t)$ and $F(s)$ are called a pair of Laplace transform.

Laplace transform of derivatives

$(i)\ \mathcal{L}[f'(t)] = sF(s) - f(0)$ (2)

$(ii) \mathcal{L}[f''(t)] = s^2F(s) - sf(0) - f'(0)$ (3)

2.2 Elzaki Transform

Definition: Given a function $f(t)$ defined for all $t \geq 0$, Elzaki transform denoted by E(.) of this function is defined as following :

$$E[f(t)] = T(v) = v\int_0^\infty f(t)\, e^{\frac{-t}{v}} dt \quad , \qquad k_1 \leq v \leq k_2 \qquad (4)$$

For all values of v , for which the improper integral converges

Elzaki transform of derivatives

$$(i)\ E[f'(t)] = \frac{T(v)}{v} - v\,f(0) \tag{5}$$

$$(ii)\ E[f''(t)] = \frac{T(v)}{v^2} - f(0) - vf'(0) \tag{6}$$

3.Applications

Example3.1. Solve the second order differential equation

$$y'' - 6y' + 9y = t^2e^{3t} \tag{7}$$

With the initial conditions

$$y(0) = 2, \quad y'(0) = 6 \tag{8}$$

Solution:

1.Applying the Laplace transform of both sides of Eq. (7) ,using (8) we get

$$s^2Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] + 9Y(s) = \frac{2}{(s-3)^3}$$

And simplifying we get

$$Y(s) = \frac{2}{s-3} + \frac{2}{(s-3)^5} \tag{9}$$

The inverse Laplace transform of Eq. (9) is simply obtained as the following

$$y(t) = 2e^{3t} + \frac{1}{12}t^4e^{3t} \tag{10}$$

2.Applying the Elzaki transform of both sides of Eq. (7)

$$E(y'') - 6E(y') + 9E(y) = E(t^2e^{3t})$$

Using (8) and simplifying give

$$T(v) = \frac{v^6}{(1-6v+9v^2)(1-3v)^3} = v^2[\frac{v^4}{(1-3v)^5}]$$

Then by partial fractions we get

$$T(v) = 2\,\frac{v^2}{1-3v} + \frac{v^6}{(1-3v)^5} \tag{11}$$

By using the inverse Elzaki transform , we have

$$y(t) = 2e^{3t} + \frac{1}{12}t^4e^{3t} \tag{12}$$

Example 3.2.Solve the second order differential equation

$$y'' + 4y' + 6y = 1 + e^{-t} \tag{13}$$

Subject to

$$y(0) = y'(0) = 0 \tag{14}$$

Solution:

1.Applying the Laplace transform of both sides of Eq. (13) , using (14) we get

$$s^2Y(s) - sy(0) - y'(0) + 4[sY(s) - y(0)] + 6Y(s) = \frac{1}{s} + \frac{1}{1+s}$$

And simplifying give

$$Y(s) = \frac{1}{3(s+1)} - \frac{\frac{s}{2} + \frac{5}{3}}{s^2 + 4s + 6} + \frac{1}{6s}$$

Then we get

$$Y(s) = \frac{\frac{1}{6}}{s} + \frac{\frac{1}{3}}{s+1} - \frac{1}{2}\frac{s+2}{(s+2)^2 + 2} - \frac{2}{3}\frac{1}{(s+2)^2 + 2} \tag{15}$$

Proceeding as before , we obtain

$$y(t) = \frac{1}{6} + \frac{1}{3}e^{-t} - \frac{1}{2}e^{-2t}\cos(\sqrt{2}t) - \frac{\sqrt{2}}{3}e^{-2t}\sin(\sqrt{2}t) \tag{16}$$

2. Applying the Elzaki transform of both sides of Eq. (13) we get

$$E(y'') + 4E(y') + 6E(y) = E(1) + E(e^{-t})$$

Using (14) and simplifying give

$$T(v) = \frac{v^2}{6} + \frac{v^2}{36}\left(\frac{2v^2 - 5v - 1}{(1+v)\left(\left(v + \frac{1}{3}\right)^2 + \frac{1}{18}\right)}\right)$$

Then by partial fractions we get

$$T(v) = \frac{v^2}{6} + \frac{1}{3}\frac{v^2}{1+v} - \frac{1}{2}\frac{(1+2v)v^2}{(1+2v)^2 + 2v^2} - \frac{2}{3}\frac{v^3}{(1+2v)^2 + 2v^2} \tag{17}$$

And therefore by the inverse Elzaki transform, we obtain

$$y(t) = \frac{1}{6} + \frac{1}{3}e^{-t} - \frac{1}{2}e^{-2t}\cos(\sqrt{2}t) - \frac{\sqrt{2}}{3}e^{-2t}\sin(\sqrt{2}t) \tag{18}$$

Example 3.3. Solve the second order differential equation

$$y'' + 16y = cos4t \tag{19}$$

Subject to

$$y(0) = 0\,, y'(0) = 1 \tag{20}$$

Solution:

1.Applying the Laplace transform of both sides of Eq. (19) , using (20) we get

$$(s^2 + 16)Y(s) = 1 + \frac{s}{s^2 + 16}$$

And simplifying give

$$Y(s) = \frac{1}{s^2 + 16} + \frac{s}{(s^2 + 16)^2} \tag{21}$$

By using the inverse Laplace transform , we have

$$y(t) = \frac{1}{4}\sin(4t) + \frac{1}{8}t\sin(4t) \tag{22}$$

2. Applying the Elzaki transform of both sides of Eq. (19)

$$E(y'') + 16E(y) = E(cos4t)$$

Using (20) and simplifying give

$$T(v) + 16v^2T(v) = \frac{v^4}{1 + 16v^2} + v^3$$

Then we get

$$T(v) = \frac{v^4}{(1 + 16v^2)^2} + \frac{v^3}{(1 + 16v^2)} \tag{23}$$

Proceeding as before , we obtain

$$y(t) = \frac{1}{4}sin(4t) + \frac{1}{8}t\,sin(4t) \tag{24}$$

Example 3.4. Solve the following system of ordinary differential equations

$$2\frac{dx}{dt} + \frac{dy}{dt} - y = t \tag{25}$$

$$\frac{dx}{dt} + \frac{dy}{dt} = t^2 \tag{26}$$

Subject to

$$x(0) = 1, y(0) = 0 \tag{27}$$

Where $x = x(t), y = y(t)$

Solution: 1. If $X(s)=\mathcal{L}[x(t)]$, $Y(s)=\mathcal{L}[y(t)]$, then after applying the Laplace transform of both sides of Eq. (25) and of Eq. (26) , using (27) we get

$$2[sX(s)-x(0)]+sY(s)-y(0)-Y(s)=\frac{1}{s^2}$$

$$sX(s)-x(0)+sY(s)-y(0)=\frac{2}{s^3}$$

And simplifying give

$$X(s)=\frac{-4}{s}+\frac{5}{s^2}-\frac{4}{s^3}+\frac{2}{s^4}+\frac{5}{s+1} \qquad (28)$$

$$Y(s)=\frac{5}{s}-\frac{5}{s^2}+\frac{4}{s^3}-\frac{5}{s+1} \qquad (29)$$

By using the inverse Laplace transform of Eq. (28) and of Eq. (29) , we obtain

$$x(t)=-4+5t-2t^2+\frac{t^3}{3}+5e^{-t} \qquad (30)$$

$$y(t)=5-5t+2t^2-5e^{-t} \qquad (31)$$

2.Applying the Elzaki transform of both sides of Eq. (25) and of Eq. (26) , using (27) we get

$$2\left[\frac{E[x(t)]}{v}-v\right]+\frac{E[y(t)]}{v}-E[y(t)]=v^3$$

$$\frac{E[x(t)]}{v}-v+\frac{E[y(t)]}{v}=2v^4$$

And simplifying give

$$E[x(t)]=-4v^2+5v^3-4v^4+2v^5+\frac{5v^2}{1+v} \qquad (32)$$

$$E[y(t)]=5v^2-5v^3+4v^4-\frac{5v^2}{1+v} \qquad (33)$$

By using the inverse Elzaki transform of Eq. (32) and of Eq. (33) , we obtain

$$x(t)=-4+5t-2t^2+\frac{t^3}{3}+5e^{-t} \qquad (34)$$

$$y(t)=5-5t+2t^2-5e^{-t} \qquad (35)$$

Example 3.5. Solve the following system of ordinary differential equations

$$\frac{d^2x}{dt^2} + 10x - 4y = 0 \tag{36}$$

$$-4x + \frac{d^2y}{dt^2} + 4y = 0 \tag{37}$$

Subject to

$$x(0) = 0, x'(0) = 1, y(0) = 0, y'(0) = -1 \tag{38}$$

Where $x = x(t), y = y(t)$

Solution: 1. If $X(s) = \mathcal{L}[x(t)]$, $Y(s) = \mathcal{L}[y(t)]$, then after applying the Laplace transform of both sides of Eq. (36) and of Eq. (37) , using (38) we get

$$s^2X(s) - sx(0) - x'(0) + 10X(s) - 4Y(s) = 0$$

$$-4X(s) + s^2Y(s) - sy(0) - y'(0) + 4Y(s) = 0$$

And simplifying give

$$X(s) = \frac{-1}{5}\frac{1}{(s^2+2)} + \frac{6}{5}\frac{1}{(s^2+12)} \tag{39}$$

$$Y(s) = \frac{-2}{5}\frac{1}{(s^2+2)} - \frac{3}{5}\frac{1}{(s^2+12)} \tag{40}$$

By using the inverse Laplace transform of Eq. (39) and of Eq. (40) , we obtain

$$x(t) = \frac{-\sqrt{2}}{10}sin(\sqrt{2}\,t) + \frac{\sqrt{3}}{5}sin(2\sqrt{3}\,t) \tag{41}$$

$$y(t) = \frac{-\sqrt{2}}{5}sin(\sqrt{2}\,t) + \frac{\sqrt{3}}{10}sin(2\sqrt{3}\,t) \tag{42}$$

2.Applying the Elzaki transform of both sides of Eq. (36) and of Eq. (37) , using (38) we get

$$\frac{E[x(t)]}{v^2} - v + 10E[x(t)] - 4E[y(t)] = 0$$

$$-4E[x(t)] + \frac{E[y(t)]}{v^2} + v + 4E[y(t)] = 0$$

Then we get

$$E[x(t)] = \frac{v^3}{(1+2v^2)(1+12v^2)}$$

$$E[y(t)] = \frac{-v^3-6v^5}{(1+2v^2)(1+12v^2)}$$

Then by partial fractions we get.

$$E[x(t)] = \frac{-1}{5}\frac{v^3}{(1+2v^2)} + \frac{6}{5}\frac{v^3}{(1+12v^2)} \quad (43)$$

$$E[y(t)] = \frac{-2}{5}\frac{v^3}{(1+2v^2)} - \frac{3}{5}\frac{v^3}{(1+12v^2)} \quad (44)$$

By using the inverse Elzaki transform of Eq. (43) and of Eq. (44) ,we obtain

$$x(t) = \frac{-\sqrt{2}}{10} sin(\sqrt{2}\, t) + \frac{\sqrt{3}}{5} sin(2\sqrt{3}\, t) \quad (45)$$

$$y(t) = \frac{-\sqrt{2}}{5} sin(\sqrt{2}\, t) + \frac{\sqrt{3}}{10} sin(2\sqrt{3}\, t) \quad (46)$$

4.conclusion

The main objective of this paper is to conduct a comparison between Laplace transform and Elzaki transform in solving some ODE's and systems of ODE's . The two methods are powerful and efficient .However using Elzaki transform to solve such problems requires more computational work if compared with the solution by using the Laplace transform.

Appendix

Elzaki transform and Laplace transform of some functions

$f(t)$	$E[f(t)] = T(v)$	$\mathcal{L}[f(t)] = F(s)$
1	v^2	$\frac{1}{s}$
t	v^3	$\frac{1}{s^2}$
t^n	$n!\ v^{n+2}$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{v^2}{1-av}$	$\frac{1}{s-a}$
te^{at}	$\frac{v^3}{(1-av)^2}$	$\frac{1}{(s-a)^2}$
$\frac{t^{n-1}e^{at}}{(n-1)!}, n = 1,2,3, \dots$	$\frac{v^{n+1}}{(1-av)^n}$	$\frac{1}{(s-a)^n}$
$\sin at$	$\frac{av^3}{1+a^2v^2}$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{v^2}{1+a^2v^2}$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{av^3}{1-a^2v^2}$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{av^2}{1-a^2v^2}$	$\frac{s}{s^2-a^2}$
$e^{at}\sin bt$	$\frac{bv^3}{(1-av)^2+b^2v^2}$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos bt$	$\frac{(1-av)v^2}{(1-av)^2+b^2v^2}$	$\frac{s}{(s-a)^2+b^2}$

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