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Convexity Preserving Integral Operator

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Abstract
The purpose of this paper, is to determine conditions of an integral operator$F(f;\ g)(z)$ given by (1) to be convex.
Key Words: and phrases: Analytic function, integral operator, convex function, close-to-convex function.

الملخص

في هذه الورقة البحثية تم عرض كل التعاريف و المفاهيم الاساسية والنظريات المساعدة التي تمكن من خلالها الى الوصول
الى نتائج هده الورقة حيث تمت دراسة بعض الخواص الهندسية للمؤثر التكاملي المعرف في الصيغة الاتية

$$F(z) = \frac{\alpha+1}{z^{\alpha}} \int_0^z (f(t)e^{g(t)})^{\alpha} dt.$$

حيث ان كل من الدوال المعرفة في المؤثر هي دوال مركبة معرفة على دوال تحليلية حول دائرة الوحدة في المستوى المركب.

Introduction

Let U be the unit disk of the complex plane:
$U = \{z : |z| < 1\}$
Let $H(U)$ denote the class of analytic functions in U. Also let

$$A_n = \{f \in H(U), f(z) = z + a_n + a_{n+1}z^{n+1} + \ldots z \in U\}$$

with $A_1 = A$,

$$K = \left\{ f \in A, Re\frac{zf''(z)}{f'(z)} + 1 > 0, \qquad z \in U \right\}$$

denote the class of convex functions in U,

$$C = \left\{ f \in A, \exists\varphi \in K, Re\frac{f'(z)}{\varphi(z)} > 0, \qquad z \in U \right\}$$

denote the class of close-to-convex functions.
In order to prove our main result, we need the following lemma.

Lemma 1.1. [9] IfP is an analytic function in U, with $ReP\,(z) > 0$ and if$P$ satisfies

$$Re\left[P(z) + \frac{zP'(z)}{P(z)}\right] > 0, \quad z \in U,$$

then $Re\,P(z) > 0, z \in U$.

Let α be a complex number, with $Re\,\alpha > 1$ and $f, g \in H(U)$. Consider the integral operator $I : H(U) \to H(U)$ denote by $F(z) = I(f;\, g)(z)$ where

$$F(z) = \frac{\alpha+1}{z^{\alpha}} \int_0^z (f(t)e^{g(t)})^{\alpha} dt. \qquad (1)$$

Many researchers have investigated many properties of integral operators forexample, Ch.Orose and G.I.Oros [10] proved that the operator

$$F(z) = \frac{1}{|g(z)|^c} \int_0^z f(w)g(w)^{c-1}g'(w)dw \quad z \in U, f \in H(U),$$

where $g \in H(U)$,with $g(0) = 0;\; g'(0) \neq 0$ and $g(z) \neq 0$, for $0 < |z| < 1$ preserves the convexity with g satisfies the conditions

$$Re\frac{czg'(z)}{g(z)} > 0$$

and

$$Re\left[1 + \frac{zg''(z)}{g'(z)}\right] > Re(c+1)\frac{zg'(z)}{g(z)}.$$

It is well-known that in particular case $g(z) = z$ and $c = 1$, Libera [3] proved that the integral operator I preserves the starlikrness,the convexity and the close-to convexity.This remarkable result was extended by many other authors (see, forexample [1],[2],[4],[5],[6],[7]).In the case $c = 1$, sufficient conditions on the function$g$ such that I is a convexity-preserving operator were given in [8].

In [9] the author shows that if g satisfies the condition

$$Re\frac{czg'(z)}{g(z)} > 0$$

in U and if the integral operator I preserves the convexity, then I also preserves the close-to-convexity.

In this paper we will show that if F satisfies the conditionsn

$$Re\frac{F(z)}{zF'(z)} > 0$$

and

$$Re\left(\frac{\alpha F(z)}{zF'(z)}\right) > Re\left(\frac{\alpha^2 F(z)}{F'(z)} + (2\alpha+1)\right)$$

in U and if F preserves the close-to-convexity, the F is also preserves the convexity.

Main Result

we begin by rewrite the integral operator $I(f, g)(z)$ by

$$F(z) = I(f.g)(z) = \frac{\alpha+1}{z^{\alpha}} \int_0^z H(t)d(t), \quad z \in U \qquad (2)$$

Theorem 2.1. Let F be the integral operator defined by (2) and suppose that.

$(i) Re\frac{F(z)}{zF'(z)} > 0, z \in U, Re\alpha > 1,$

$(ii)\ Re\frac{F(z)}{zF'(z)} > Re\left(\frac{\alpha^2 F(z)}{F'(z)} + (2\alpha + 1)\right)$

$(iii)\ F(C) \subset C$

then

$F(K) \subset K.$

Proof. It is clear that $H(z) \in H(U)$ and $H(z) \neq 0$ in U.From (2) we obtain

$$F'(z)\frac{z^\alpha}{\alpha+1} + \frac{\alpha z^{\alpha+1}}{\alpha+1}F(z) = H(z), \qquad (3)$$

let $h(t) = \frac{z^{\alpha+1}}{\alpha+1}$, then (3) become

$$zF'(z)h(z) + \alpha F(z)h(z) = H(z)$$

differentiating last equation, we obtain

$$zF''(z)h(z) + zF'(z)h'(z) + F'(z)h(z) + \alpha h(z)F'(z) + \alpha h'(z)F(z) = H'(z),$$

which is equivalent to

$$F'(z)h(z)\left[\frac{zF''(z)}{F'(z)} + \frac{zh'(z)}{h(z)} + (1+\alpha) + \frac{\alpha h'(z)F(z)}{h(z)F'(z)}\right] = H'(z)$$

Let $H \in C$. Then there exists $\psi \in K$ such that

$$Re\frac{H'(z)}{\psi'(z)} > 0, \quad z \in U.$$

If we denote $\beta = I(\psi)$, then

$$\beta(z) = \frac{\alpha+1}{z^\alpha}\int_0^z \psi(t)d(t), \qquad z \in U. \qquad (4)$$

Next we prove $\beta \in K$. rentiating (4), we obtain

$$\beta'(z)h(z)\left[\frac{z\beta''(z)}{\beta'(z)} + \frac{zh'(z)}{h(z)} + (1+\alpha) + \frac{\alpha h'(z)\beta(z)}{h(z)\beta'(z)}\right] = \psi'(z). \quad (5)$$

If we let

$$P(z) = \frac{z\beta''(z)}{\beta'(z)} + \frac{zh'(z)}{h(z)} + (1+\alpha) + \frac{\alpha h'(z)\beta(z)}{h(z)\beta'(z)}, \quad z \in U \qquad (6)$$

then (5) becomes

$$\beta'(z)h(z)P(z) = \psi'(z) \qquad z \in U. \qquad (7)$$

Differentiating (7), we obtain

$$\frac{z\beta''(z)}{\beta'(z)}+\frac{zh'(z)}{h(z)}+\frac{zP'(z)}{P(z)}=\frac{z\psi''(z)}{\psi'(z)},\quad z\in U$$

which is equivalent to

$$\frac{z\beta''(z)}{\beta'(z)}+\frac{zh'(z)}{h(z)}+\frac{zP'(z)}{P(z)}+(\alpha+1)+\frac{\alpha\beta(z)h'(z)}{h(z)\beta'(z)}=\frac{z\psi''(z)}{\psi'(z)}+(\alpha+1)+\frac{\alpha\beta(z)h'(z)}{h(z)\beta'(z)},\ z\in U \quad (8)$$

Using (6) in (8), we obtain

$$P(z)+\frac{zP'(z)}{P(z)}=\frac{z\psi''(z)}{\psi'(z)}+(\alpha+1)+\frac{\alpha\beta(z)h'(z)}{h(z)\beta'(z)},\ z\in U \qquad (9)$$

Using condition (i) from hypothesis, since ψ is convex and$h=\frac{z^{\alpha+1}}{\alpha+1}$ we have

$$Re\left[P(z)+\frac{zP'(z)}{P(z)}\right]=Re\left[\frac{z\psi''(z)}{\psi'(z)}+1+\alpha+\frac{\alpha(\alpha-1)\beta(z)}{z\beta'(z)}\right],\quad z\in U \quad (10)$$

i.e

$$Re\left[P(z)+\frac{zP'(z)}{P(z)}\right]>0,\quad z\in U.$$

Letting z=0 in (10), we get

$$ReP(0)>0,\ \ z\in U.$$

We have now the conditions from the hypothesis of Lemma 1.1 and applying it we obtain

$$ReP(0)>0,\quad z\in U.$$

Using (6) and the condition $ReP\ (0)>0, z\in U$: We obtain

$$Re\left[\frac{z\beta''(z)}{\beta'(z)}+\frac{zh'(z)}{h(z)}+(1+\alpha)+\frac{\alpha h'(z)\beta(z)}{h(z)\beta'(z)}\right]>0$$

and using (ii) and $h=\frac{z^{\alpha+1}}{\alpha+1}$we obtain

$$Re\left[\frac{z\beta''(z)}{\beta'(z)}+1\right]>-2\alpha+1-\frac{\alpha(\alpha-1)\beta(z)}{z\beta'(z)}>0,\ \ z\in U$$

i.e

$$Re\left[\frac{z\beta''(z)}{\beta'(z)}+1\right]>0,\ \ z\in U$$

which shows that $\beta\in K$.

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