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العدد العشرون
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هيئة تحرير
مجلة التربوي

- المجلة ترحب بما يرد عليها من أبحاث وعلى استعداد لنشرها بعد التحكيم .
 - المجلة تحترم كل الاحترام آراء المحكمين وتعمل بمقتضاها .
 - كافة الآراء والأفكار المنشورة تعبر عن آراء أصحابها ولا تتحمل المجلة تبعاتها .
 - يتحمل الباحث مسؤولية الأمانة العلمية وهو المسؤول عما ينشر له .
 - البحوث المقدمة للنشر لا ترد لأصحابها نشرت أو لم تنشر .
- (حقوق الطبع محفوظة للكلية)



ضوابط النشر:

- يشترط في البحوث العلمية المقدمة للنشر أن يراعى فيها ما يأتي :
- أصول البحث العلمي وقواعده .
 - ألا تكون المادة العلمية قد سبق نشرها أو كانت جزءا من رسالة علمية .
 - يرفق بالبحث تزكية لغوية وفق أنموذج معد .
 - تعدل البحوث المقبولة وتصحح وفق ما يراه المحكمون .
 - التزام الباحث بالضوابط التي وضعتها المجلة من عدد الصفحات ، ونوع الخط ورقمه ، والفترات الزمنية الممنوحة للتعديل ، وما يستجد من ضوابط تضعها المجلة مستقبلا .

تنبيهات :

- للمجلة الحق في تعديل البحث أو طلب تعديله أو رفضه .
- يخضع البحث في النشر لأولويات المجلة وسياستها .
- البحوث المنشورة تعبر عن وجهة نظر أصحابها ، ولا تعبر عن وجهة نظر المجلة .

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On a certain class of p -valent functions with negative coefficients

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Abstract

In this paper we introduce a new class of p -valent starlike functions with negative coefficients defined by fractional derivative operator. We obtain coefficient bounds, distortion inequalities, Hadamard product, linear combinations and inclusion theorems. Also, we find extreme points and radii of close-to-convexity, starlikeness and convexity for this class. The integral preserving properties and integral means inequalities are also determined.

Keywords: multivalent (p -valent) functions, starlike functions, convex functions, close-to-convex functions, fractional derivatives, Hadamard product, integral means.

1- Introduction and Definitions

Let $A(p)$ denote the class of functions defined by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad (p \in \mathbb{N}). \quad (1.1)$$

which are analytic and multivalent (or p -valent) in the open unit disk $\mathcal{U} = \{z: |z| < 1\}$. We write $A(1) = A$. If f and g are analytic in \mathcal{U} , we say that f is subordinate to g , written symbolically as $f < g$ or $f(z) < g(z), z \in \mathcal{U}$, if there exists a Schwarz function $w(z)$ which is analytic in \mathcal{U} with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z)), z \in \mathcal{U}$.

A function $f(z) \in A(p)$ is said to be p -valent starlike of order α if $f(z)$ satisfies the condition

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad (0 \leq \alpha < p; p \in \mathbb{N}; z \in \mathcal{U}) \quad (1.2)$$

We denote by $S^*(p, \alpha)$ the class of all such functions. A function $f(z) \in A(p)$ is said to be p -valent convex of order α if $f(z)$ satisfies the condition



$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad (0 \leq \alpha < p; p \in \mathbb{N}; z \in \mathcal{U}) \quad (1.3)$$

Let $K(p, \alpha)$ denote the class of all those functions which are p -valent convex of order α . The class $S^*(p, \alpha)$ was introduced by Patil and Thakare [6], and the class $K(p, \alpha)$ was introduced by Owa [5].

Let $T(p)$ denote the subclass of $A(p)$ consisting of functions of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (a_{p+n} \geq 0, p \in \mathbb{N}, z \in \mathcal{U}). \quad (1.4)$$

We denote by $T^*(p, \alpha)$ and $C(p, \alpha)$ the classes obtained by taking intersections, respectively, of the classes $S^*(p, \alpha)$ and $K(p, \alpha)$ with the class $T(p)$. The classes $T^*(p, \alpha)$ and $C(p, \alpha)$ were introduced by Owa [5]. In particular, the classes $T^*(1, \alpha) = T^*(\alpha)$ and $C(1, \alpha) = C(\alpha)$ when $p = 1$ were studied by Silverman [8].

Let the functions $f_i(z)$, ($i = 1, 2$) be defined by

$$f_i(z) = z^p - \sum_{n=1}^{\infty} a_{p+n,i} z^{p+n}, \quad (a_{p+n,i} \geq 0; p \in \mathbb{N}). \quad (1.5)$$

The Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{n=1}^{\infty} a_{p+n,1} a_{p+n,2} z^{p+n} \quad (1.6)$$

Definition 1.1.[1,2,10]. Let $0 \leq \lambda < 1$, and $\mu, \eta \in \mathbb{R}$, the fractional derivative operator $J_{0,z}^{\lambda, \mu, \eta}$ is defined in terms of Gauss's hypergeometric function ${}_2F_1$ as follows

$$J_{0,z}^{\lambda, \mu, \eta} f(z) = \frac{d}{dz} \left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)} \int_0^z (z-\xi)^{-\lambda} f(\xi) {}_2F_1 \left(\mu - \lambda, 1 - \eta; 1 - \lambda; 1 - \frac{\xi}{z} \right) d\xi \right) \quad (1.7)$$

where $f(z)$ is analytic function in a simply- connected region of the z -plane containing the origin with the order $f(z) = O(|z|^\varepsilon)$, $z \rightarrow 0$, where $\varepsilon > \max\{0, \mu - \eta\} - 1$, and the multiplicity of $(z - \xi)^{-\lambda}$ is removed by requiring $\log(z - \xi)$ to be real when $z - \xi > 0$.

We note that, $J_{0,z}^{\lambda, \lambda, \eta} f(z) = D_z^\lambda f(z)$, $0 \leq \lambda < 1$ is the fractional derivative operator considered by Owa [4].

The fractional derivative operator $M_{0,z}^{\lambda, \mu, \eta, p} f(z)$ of a function $f(z)$ in $A(p)$ is defined by



$$M_{0,z}^{\lambda,\mu,\eta,p} f(z) = \frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\eta)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)} z^\mu J_{0,z}^{\lambda,\mu,\eta} f(z) \quad (1.8)$$

$$(\lambda \geq 0; \mu < p+1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbb{N})$$

The operator $M_{0,z}^{\lambda,\mu,\eta,p} f(z)$ was studied by Amsheri and Zharkova [1]. (see also [10]).

Recently, Zayed et al. [10] introduced the operator $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)$ of a function $f(z)$ in $A(p)$ for $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\delta \geq 0$ and defined by:

$$\begin{aligned} N_{0,z}^{0,\lambda,\mu,\eta,\delta,p} f(z) &= M_{0,z}^{\lambda,\mu,\eta,p} f(z) \\ N_{0,z}^{1,\lambda,\mu,\eta,\delta,p} f(z) &= N_{0,z}^{\lambda,\mu,\eta,\delta,p} f(z) \\ &= (1-\delta)M_{0,z}^{\lambda,\mu,\eta,p} f(z) + \delta \frac{z}{p} \left(M_{0,z}^{\lambda,\mu,\eta,p} f(z) \right)' \end{aligned}$$

and (in general),

$$\begin{aligned} N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z) &= N_{0,z}^{\lambda,\mu,\eta,\delta,p} \left(N_{0,z}^{m-1,\lambda,\mu,\eta,\delta,p} f(z) \right) \\ &= z^p + \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p} \right)^m \gamma_n(\lambda, \mu, \eta, p) a_{p+n} z^{p+n} \end{aligned} \quad (1.9)$$

where

$$\gamma_n(\lambda, \mu, \eta, p) = \frac{(p+1)_n (p+1-\mu+\eta)_n}{(p+1-\mu)_n (p+1-\lambda+\eta)_n}, \quad (n \in \mathbb{N}) \quad (1.10)$$

Motivated essentially by aforementioned works, we introduce a new class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$ of analytic and p -valent functions $f(z)$ belonging to the class $T(p)$ by using the operator $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)$ as follows:

Definition 1.2. The function $f(z) \in T(p)$ is said to be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$ if it satisfies

$$\frac{\frac{1}{p} z \left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z) \right)'}{(1-\beta)z^p + \beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)} < \frac{1 + (1-2\alpha)z}{1-z}, \quad (z \in \mathcal{U}; p \in \mathbb{N}) \quad (1.11)$$

For $m \in \mathbb{N}_0; \lambda \geq 0; \mu < p+1; \eta > \max(\lambda, \mu) - p - 1; 0 < \beta \leq 1; 0 \leq \alpha < 1; \delta \geq 0$. The condition (1.11) is equivalent to



$$\left| \frac{\frac{\frac{1}{p}z(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z))'}{(1-\beta)z^p + \beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)} - 1}{\frac{\frac{1}{p}z(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z))'}{(1-\beta)z^p + \beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)} + 1 - 2\alpha} \right| < 1 \quad (z \in \mathcal{U}; p \in \mathbb{N}) \quad (1.12)$$

Easily we can deduce that,

$$N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z) = z^p - \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda, \mu, \eta, p) a_{p+n} z^{p+n} \quad (a_{p+n} \geq 0, p \in \mathbb{N}) \quad (1.13)$$

where $\gamma_n(\lambda, \mu, \eta, p)$ is given by (1.10). We observe that,

- 1- For $\lambda = \mu = \delta = 0$, the class $\mathcal{M}_p^{m,0,0,\eta,0}(\beta, \alpha) = \mathcal{M}_p(\beta, \alpha)$ (Selvaraj et al. [7]).
- 2- For $\lambda = \mu = \delta = 0$ and $\beta = 1$, the class $\mathcal{M}_p^{m,0,0,\eta,0}(1, \alpha) = T^*(p, \alpha)$ (Owa [5]).
- 3- For $\lambda = \mu = \delta = 0$ and $p = \beta = 1$, the class $\mathcal{M}_1^{m,0,0,\eta,0}(1, \alpha) = T^*(\alpha)$ (Silverman [8]).

In the present paper, we obtain coefficient bounds, distortion inequalities, Hadamard product, linear combinations and inclusion theorems for the functions belonging to the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$. Also, we find extreme points and radii of close-to-convexity, starlikeness and convexity for this class. Finally, we determine the integral means inequalities and a class-preserving integral operator of the form

$$F(z) = (J_{c,p}f)(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt, \quad (c > -p) \quad (1.14)$$

In order to prove our results in section 9 we shall need the following lemma.

Lemma 1.3. [3]. If the functions $f(z)$ and $g(z)$ are analytic in \mathcal{U} with $g(z) < f(z)$, then

$$\int_0^{2\pi} |f(re^{i\theta})|^\tau d\theta \leq \int_0^{2\pi} |g(re^{i\theta})|^\tau d\theta, \quad (z = re^{i\theta}, 0 < r < 1) \quad (1.15)$$

2- Coefficient Bounds

Theorem 2.1. Let the function $f(z)$ be defined by (1.4). Then $f(z)$ belongs to the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$ if and only if

$$\sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)] \gamma_n(\lambda, \mu, \eta, p) a_{p+n} \leq p(1-\alpha) \quad (2.1)$$

where $\gamma_n(\lambda, \mu, \eta, p)$ is given by (1.10).



Proof. Since $f(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$, then

$$\left| \frac{\frac{1}{p}z(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z))' - ((1-\beta)z^p + \beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z))}{\frac{1}{p}z(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z))' + (1-2\alpha)((1-\beta)z^p + \beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z))} \right| < 1. \quad (2.2)$$

It follows from (2.2) that

$$\operatorname{Re} \left\{ \frac{\sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda, \mu, \eta, p) \left[\left(\frac{p+n}{p}\right) - \beta\right] a_{p+n} z^n}{2(1-\alpha) - \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda, \mu, \eta, p) \left[\left(\frac{p+n}{p}\right) + \beta(1-2\alpha)\right] a_{p+n} z^n} \right\} < 1.$$

Choosing values of z on the real axis so that $\frac{\frac{1}{p}z(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z))'}{(1-\beta)z^p + \beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)}$ is real, and letting

$z \rightarrow 1^-$ through real axis, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda, \mu, \eta, p) \left[\left(\frac{p+n}{p}\right) - \beta\right] a_{p+n} \\ & \leq 2(1-\alpha) - \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda, \mu, \eta, p) \left[\left(\frac{p+n}{p}\right) + \beta(1-2\alpha)\right] a_{p+n} \end{aligned}$$

which gives the desired assertion (2.1). Conversely, let the inequality (2.1) holds true and let $|z| = 1$. Then we have

$$\begin{aligned} & \left| \frac{1}{p}z(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z))' - ((1-\beta)z^p + \beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)) \right| \\ & - \left| \frac{1}{p}z(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z))' + (1-2\alpha)((1-\beta)z^p + \beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)) \right| \\ & = \left| - \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda, \mu, \eta, p) \left[\left(\frac{p+n}{p}\right) - \beta\right] a_{p+n} z^{p+n} \right| \\ & - \left| 2(1-\alpha)z^p - \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda, \mu, \eta, p) \left[\left(\frac{p+n}{p}\right) + \beta(1-2\alpha)\right] a_{p+n} z^{p+n} \right| \\ & \leq 2 \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m \left[\left(\frac{p+n}{p}\right) - \alpha\beta\right] \gamma_n(\lambda, \mu, \eta, p) a_{p+n} - 2(1-\alpha) \leq 0. \end{aligned}$$

Hence by the maximum modulus theorem, $f(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$. This completes the proof.



Corollary 2.2. Let the function $f(z)$ be defined by (1.4) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$, then

$$a_{p+n} \leq \frac{p(1-\alpha)}{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)] \gamma_n(\lambda, \mu, \eta, p)}, \quad (p, n \in \mathbb{N}) \quad (2.3)$$

where $\gamma_n(\lambda, \mu, \eta, p)$ is given by (1.10). The result (2.3) is sharp for a function of the form:

$$f(z) = z^p - \frac{p(1-\alpha)}{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)] \gamma_n(\lambda, \mu, \eta, p)} z^{p+n}, \quad (p, n \in \mathbb{N}) \quad (2.4)$$

Remark 1. Letting $p = 1, \lambda = \mu = \delta = 0$ and $\beta = 1$ in Theorem 2.1 and Corollary 2.2 respectively, we obtain the results were proved by Silverman [8].

3- Distortion Inequalities

Theorem 3.1. Let $\lambda, \mu, \eta \in \mathbb{R}$, such that

$$\lambda \geq 0, \mu < p + 1, \eta \geq \lambda \left(1 - \frac{p+2}{\mu}\right), \delta \geq 0, m \in \mathbb{N}_0, 0 < \beta \leq 1, 0 \leq \alpha < 1 \text{ and } p \in \mathbb{N}. \quad (3.1)$$

Let the function $f(z)$ defined by (1.4) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$. Then

$$|f(z)| \geq |z|^p - \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)} |z|^{p+1}, \quad (3.2)$$

$$|f(z)| \leq |z|^p + \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)} |z|^{p+1}, \quad (3.3)$$

$$|f'(z)| \geq p|z|^{p-1} - \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1-\mu+\eta)} |z|^p, \quad (3.4)$$

and

$$|f'(z)| \leq p|z|^{p-1} + \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1-\mu+\eta)} |z|^p, \quad (3.5)$$

for $z \in \mathcal{U}$. The estimates for $|f(z)|$ and $|f'(z)|$ are sharp.

Proof. We observe that the function $\gamma_n(\lambda, \mu, \eta, p)$ defined by (1.10) satisfy the inequality $\gamma_n(\lambda, \mu, \eta, p) \leq \gamma_{n+1}(\lambda, \mu, \eta, p)$, $\forall n \in \mathbb{N}$, provided that $\eta \geq \lambda \left(1 - \frac{p+2}{\mu}\right)$.



Thereby, showing that $\gamma_n(\lambda, \mu, \eta, p)$ is non-decreasing. Thus under the conditions stated in (3.1), we have

$$0 < \frac{(p+1)(p+1-\mu+\eta)}{(p+1-\mu)(p+1-\lambda+\eta)} = \gamma_1(\lambda, \mu, \eta, p) \leq \gamma_n(\lambda, \mu, \eta, p) \quad \forall n \in \mathbb{N}, \quad (3.6)$$

for $f(z) \in \mathcal{M}_p^{m, \lambda, \mu, \eta, \delta}(\beta, \alpha)$, in view of Theorem 2.1, we have

$$\begin{aligned} & \frac{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)}{(p+1-\mu)(p+1-\lambda+\eta)} \sum_{n=1}^{\infty} a_{p+n} \\ & \leq \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)] \gamma_n(\lambda, \mu, \eta, p) a_{p+n} \leq p(1-\alpha), \end{aligned} \quad (3.7)$$

which gives

$$\sum_{n=1}^{\infty} a_{p+n} \leq \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)} \quad (3.8)$$

Consequently, we obtain

$$\begin{aligned} |f(z)| & \geq |z|^p - |z|^{p+1} \sum_{n=1}^{\infty} a_{p+n} \\ & \geq |z|^p - \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)} |z|^{p+1} \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} |f(z)| & \leq |z|^p + |z|^{p+1} \sum_{n=1}^{\infty} a_{p+n} \\ & \leq |z|^p + \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)} |z|^{p+1} \end{aligned} \quad (3.10)$$

which prove the assertions (3.2) and (3.3) of Theorem 3.1. Furthermore, from Theorem 2.1, we note that

$$\sum_{n=1}^{\infty} (p+n) a_{p+n} \leq \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1-\mu+\eta)}. \quad (3.11)$$

Thus, we have



$$|f'(z)| \geq p|z|^{p-1} - |z|^p \sum_{n=1}^{\infty} (p+n)a_{p+n} \quad (3.12)$$

and

$$|f'(z)| \leq p|z|^{p-1} + |z|^p \sum_{n=1}^{\infty} (p+n)a_{p+n} . \quad (3.13)$$

On using (3.12), (3.13) and (3.11), we arrive at the desired results (3.4) and (3.5).

Finally, we can prove that the estimates for $|f(z)|$ and $|f'(z)|$ are sharp by taking the function

$$f(z) = z^p - \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)} z^{p+1} \quad (3.14)$$

Corollary 3.2. Let the function $f(z)$ be defined by (1.4) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$.

Then $f(z)$ is included in a disk with centre at the origin and radius r_1 given by

$$r_1 = 1 + \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)} \quad (3.15)$$

$f'(z)$ is included in a disk with centre at the origin and radius r_2 given by

$$r_2 = p + \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1-\mu+\eta)} \quad (3.16)$$

4- Hadamard Product

Theorem 4.1. Let $\lambda \geq 0$; $\mu < p+1$; $\eta \geq \lambda \left(1 - \frac{p+2}{\mu}\right)$; $\delta \geq 0$; $m \in \mathbb{N}_0$; $0 < \beta \leq 1$; $0 \leq \alpha < 1$; $\lambda, \mu, \eta \in \mathbb{R}$ and $p \in \mathbb{N}$, and let the functions $f_i(z)$ ($i = 1, 2$) defined by (1.5) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Then $(f_1 * f_2)(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\sigma)$ where

$$\sigma \leq \inf_{n \in \mathbb{N}} \left\{ \frac{v(n) - p(p+n)(1-\alpha)^2}{v(n) - p^2\beta(1-\alpha)^2} \right\} \quad (4.1)$$

where

$$v(n) = \left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]^2 \gamma_n(\lambda, \mu, \eta, p) \quad (4.2)$$

Proof. It suffices to prove that



$$\sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\sigma)]\gamma_n(\lambda, \mu, \eta, p)}{p(1-\sigma)} a_{p+n,1} a_{p+n,2} \leq 1. \quad (4.3)$$

Since

$$\sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda, \mu, \eta, p)}{p(1-\alpha)} a_{p+n,1} \leq 1, \quad (4.4)$$

and

$$\sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda, \mu, \eta, p)}{p(1-\alpha)} a_{p+n,2} \leq 1. \quad (4.5)$$

By the Cauchy – Schwarz inequality, we have

$$\sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda, \mu, \eta, p)}{p(1-\alpha)} \sqrt{a_{p+n,1} a_{p+n,2}} \leq 1 \quad (4.6)$$

Thus, we need to find the largest σ such that

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\sigma)]\gamma_n(\lambda, \mu, \eta, p)}{p(1-\sigma)} a_{p+n,1} a_{p+n,2} \\ & \leq \sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda, \mu, \eta, p)}{p(1-\alpha)} \sqrt{a_{p+n,1} a_{p+n,2}}, \end{aligned} \quad (4.7)$$

or, equivalently, that

$$\sqrt{a_{p+n,1} a_{p+n,2}} \leq \frac{[n+p(1-\beta\alpha)](1-\sigma)}{[n+p(1-\beta\sigma)](1-\alpha)}. \quad (4.8)$$

In view of (4.6), it is sufficient to find the largest σ such that

$$\frac{p(1-\alpha)}{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda, \mu, \eta, p)} \leq \frac{[n+p(1-\beta\alpha)](1-\sigma)}{[n+p(1-\beta\sigma)](1-\alpha)}, \quad (n \in \mathbb{N}). \quad (4.9)$$

The inequality (4.9) yields

$$\sigma \leq \left\{ \frac{v(n) - p(p+n)(1-\alpha)^2}{v(n) - p^2\beta(1-\alpha)^2} \right\}$$

where



$$v(n) = \left(\frac{p + \delta n}{p}\right)^m [n + p(1 - \beta\alpha)]^2 \gamma_n(\lambda, \mu, \eta, p),$$

and this the inequality gives the required result.

Corollary 4.2. For $f_i(z)$ ($i = 1, 2$) as Theorem 4.1, we have

$$h(z) = z^p - \sum_{n=1}^{\infty} \sqrt{a_{p+n,1} a_{p+n,2}} z^{p+n}, \quad (p \in \mathbb{N}) \quad (4.10)$$

belongs to the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$.

Proof. The result follows from the inequality (4.6).

Theorem 4.3. Let the function $f(z)$ defined by (1.4) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$. Also let

$$g(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n}, \quad (|b_{p+n}| \leq 1, p \in \mathbb{N}) \quad (4.11)$$

Then $(f * g)(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$.

Proof. Since

$$\begin{aligned} & \sum_{n=1}^{\infty} \left(\frac{p + \delta n}{p}\right)^m [n + p(1 - \beta\alpha)] \gamma_n(\lambda, \mu, \eta, p) |a_{p+n} b_{p+n}| \\ &= \sum_{n=1}^{\infty} \left(\frac{p + \delta n}{p}\right)^m [n + p(1 - \beta\alpha)] \gamma_n(\lambda, \mu, \eta, p) a_{p+n} |b_{p+n}| \\ &\leq \sum_{n=1}^{\infty} \left(\frac{p + \delta n}{p}\right)^m [n + p(1 - \beta\alpha)] \gamma_n(\lambda, \mu, \eta, p) a_{p+n} \\ &\leq p(1 - \alpha). \end{aligned}$$

By Theorem 2.1, it follows that

$$(f * g)(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha).$$

5- Linear Combinations and Inclusion Theorems

We shall prove that the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$ is closed under arithmetic mean and under convex linear combinations.

Theorem 5.1. Let the functions $f_i(z)$ ($i = 1, 2, \dots, m$) defined by (1.5) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$. Then the function



$$h(z) = z^p - \frac{1}{m} \sum_{n=1}^{\infty} \left(\sum_{i=1}^m a_{p+n,i} \right) z^{p+n} \quad (5.1)$$

is also in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$.

Proof. Since $f_i(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$, by using Theorem 2.1, we have

$$\sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda, \mu, \eta, p)}{p(1-\alpha)} a_{p+n,i} \leq 1, \quad (i = 1, 2, 3, \dots, m) \quad (5.2)$$

so,

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda, \mu, \eta, p)}{p(1-\alpha)} \left(\frac{1}{m} \sum_{i=1}^m a_{p+n,i} \right) \\ &= \frac{1}{m} \sum_{i=1}^m \sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda, \mu, \eta, p)}{p(1-\alpha)} a_{p+n,i} \leq 1 \end{aligned} \quad (5.3)$$

which shows that $h(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$, and the proof of Theorem 5.1 is completed.

Theorem 5.2. Let $\lambda \geq 0$; $\mu < p + 1$; $\eta \geq \lambda \left(1 - \frac{p+2}{\mu}\right)$; $\delta \geq 0$; $m \in \mathbb{N}_0$; $0 < \beta \leq 1$; $0 \leq \alpha < 1$; $\lambda, \mu, \eta \in \mathbb{R}$ and $p \in \mathbb{N}$, and let the functions $f_i(z)$ ($i = 1, 2$) defined by (1.5) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$. Then the function

$$h(z) = z^p - \sum_{n=1}^{\infty} (a_{p+n,1}^2 + a_{p+n,2}^2) z^{p+n}, \quad (p \in \mathbb{N}) \quad (5.4)$$

belongs to the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \sigma)$, where

$$\sigma \leq \inf_{n \in \mathbb{N}} \left\{ \frac{v(n) - 2p(p+n)(1-\alpha)^2}{v(n) - 2p^2\beta(1-\alpha)^2} \right\} \quad (5.5)$$

Where $v(n)$ given by (4.2).

Proof. By virtue of Theorem 2.1, we obtain

$$\sum_{n=1}^{\infty} \left\{ \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda, \mu, \eta, p)}{p(1-\alpha)} \right\}^2 a_{p+n,1}^2$$



$$\leq \left\{ \sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda, \mu, \eta, p)}{p(1-\alpha)} a_{p+n,1} \right\}^2 \leq 1 \quad (5.6)$$

and

$$\sum_{n=1}^{\infty} \left\{ \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda, \mu, \eta, p)}{p(1-\alpha)} \right\}^2 a_{p+n,2}^2$$
$$\leq \left\{ \sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda, \mu, \eta, p)}{p(1-\alpha)} a_{p+n,2} \right\}^2 \leq 1. \quad (5.7)$$

It follows from (5.6) and (5.7) that

$$\sum_{n=1}^{\infty} \frac{1}{2} \left\{ \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda, \mu, \eta, p)}{p(1-\alpha)} \right\}^2 (a_{p+n,1}^2 + a_{p+n,2}^2) \leq 1. \quad (5.8)$$

Therefore, we need to find the largest σ such that

$$\sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\sigma)]\gamma_n(\lambda, \mu, \eta, p)}{p(1-\sigma)} (a_{p+n,1}^2 + a_{p+n,2}^2) \leq 1. \quad (5.9)$$

Thus, it is sufficient to show that

$$\frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\sigma)]\gamma_n(\lambda, \mu, \eta, p)}{p(1-\sigma)}$$
$$\leq \frac{1}{2} \left\{ \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda, \mu, \eta, p)}{p(1-\alpha)} \right\}^2, \quad (n \geq 1). \quad (5.10)$$

The inequality (5.10) yields

$$\sigma \leq \left\{ \frac{v(n) - 2p(p+n)(1-\alpha)^2}{v(n) - 2p^2\beta(1-\alpha)^2} \right\}$$

where $v(n)$ given by (4.2), and this inequality gives the required result.

Theorem 5.3. The class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$ is convex.



Proof. Suppose that the functions $f_i(z)(i = 1,2)$ defined by (1.5) are in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$. Then it is sufficient to show that the function

$$h(z) = \xi f_1(z) + (1 - \xi)f_2(z) \quad , (0 \leq \xi \leq 1) \quad (5.11)$$

or, equivalently

$$h(z) = z^p - \sum_{n=1}^{\infty} \{\xi a_{p+n,1} + (1 - \xi)a_{p+n,2}\} z^{p+n} \quad , (0 \leq \xi \leq 1) \quad (5.12)$$

is also in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$.

Now, from our hypothesis and Theorem 2.1, it follows readily that

$$\sum_{n=1}^{\infty} \left(\frac{p + \delta n}{p}\right)^m [n + p(1 - \beta\alpha)] \gamma_n(\lambda, \mu, \eta, p) (\xi a_{p+n,1} + (1 - \xi)a_{p+n,2}) \leq p(1 - \alpha)$$

which evidently proves Theorem 5.3.

6- Extreme Points

Theorem 6.1. Let

$$f_p(z) = z^p, \quad (p \in \mathbb{N}) \quad (6.1)$$

and

$$f_{p+n}(z) = z^p - \frac{p(1 - \alpha)}{\left(\frac{p + \delta n}{p}\right)^m [n + p(1 - \beta\alpha)] \gamma_n(\lambda, \mu, \eta, p)} z^{p+n}, \quad (p, n \in \mathbb{N}). \quad (6.2)$$

Then $f(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$ if and only if it can be expressed in the form:

$$f(z) = \sum_{n=0}^{\infty} \varepsilon_{p+n} f_{p+n}(z) \quad (6.3)$$

where

$$\varepsilon_{p+n} \geq 0 \quad , \quad \sum_{n=0}^{\infty} \varepsilon_{p+n} = 1 \quad (6.4)$$

Proof. Let

$$f(z) = \sum_{n=0}^{\infty} \varepsilon_{p+n} f_{p+n}(z)$$



$$= z^p - \sum_{n=1}^{\infty} \frac{p(1-\alpha)}{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)] \gamma_n(\lambda, \mu, \eta, p)} \varepsilon_{p+n} z^{p+n}. \quad (6.5)$$

Then, in view of (6.4), it follows that

$$\sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)] \gamma_n(\lambda, \mu, \eta, p)}{p(1-\alpha)} \left\{ \frac{p(1-\alpha)}{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)] \gamma_n(\lambda, \mu, \eta, p)} \varepsilon_{p+n} \right\} \\ = \sum_{n=1}^{\infty} \varepsilon_{p+n} = 1 - \varepsilon_p \leq 1. \quad (6.6)$$

So, by Theorem 2.1, the function $f(z)$ belongs to the class $\mathcal{M}_p^{m, \lambda, \mu, \eta, \delta}(\beta, \alpha)$.

Conversely, let the function $f(z)$ defined by (1.4) belongs to the class $\mathcal{M}_p^{m, \lambda, \mu, \eta, \delta}(\beta, \alpha)$, then

$$a_{p+n} \leq \frac{p(1-\alpha)}{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)] \gamma_n(\lambda, \mu, \eta, p)}, \quad (p, n \in \mathbb{N}). \quad (6.7)$$

Setting

$$\varepsilon_{p+n} = \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)] \gamma_n(\lambda, \mu, \eta, p)}{p(1-\alpha)} a_{p+n}, \quad (p, n \in \mathbb{N}), \quad (6.8)$$

and

$$\varepsilon_p = 1 - \sum_{n=1}^{\infty} \varepsilon_{p+n} \quad (6.9)$$

we can see that $f(z)$ can be expressed in the form (6.3). This completes the proof of Theorem 6.1.

Corollary 6.2. The extreme points of the class $\mathcal{M}_p^{m, \lambda, \mu, \eta, \delta}(\beta, \alpha)$ are the functions $f_p(z)$ and $f_{p+n}(z)$ given by (6.1) and (6.2) respectively.

7- Radii of Close-to-convexity, Starlikeness and Convexity

Theorem 7.1. Let the function $f(z)$ defined by (1.4) be in the class $\mathcal{M}_p^{m, \lambda, \mu, \eta, \delta}(\beta, \alpha)$. Then $f(z)$ is p -valently close-to-convex of order σ ($0 \leq \sigma < p$) in the disk $|z| < r_3$ where



$$r_3 = \inf_{n \in \mathbb{N}} \left\{ \frac{(p - \sigma) \left(\frac{p + \delta n}{p} \right)^m [n + p(1 - \beta\alpha)] \gamma_n(\lambda, \mu, \eta, p)}{p(1 - \alpha)(p + n)} \right\}^{1/n} \quad (7.1)$$

and $\gamma_n(\lambda, \mu, \eta, p)$ is given by (1.10). The result is sharp with the external function $f(z)$ given by (2.4).

Proof . It suffices to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \sigma \quad , (|z| < r_3) \quad (7.2)$$

Indeed we have

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq \sum_{n=1}^{\infty} (p + n) a_{p+n} |z|^n \quad (7.3)$$

Hence (7.2) is true if

$$\sum_{n=1}^{\infty} (p + n) a_{p+n} |z|^n \leq p - \sigma,$$

or

$$\sum_{n=1}^{\infty} \frac{(p + n)}{(p - \sigma)} a_{p+n} |z|^n \leq 1. \quad (7.4)$$

By Theorem 2.1, (7.4) is true if

$$\frac{(p + n)}{(p - \sigma)} |z|^n \leq \frac{\left(\frac{p + \delta n}{p} \right)^m [n + p(1 - \beta\alpha)] \gamma_n(\lambda, \mu, \eta, p)}{p(1 - \alpha)} \quad , \quad (n \in \mathbb{N}). \quad (7.5)$$

Solving (7.5) for $|z|$, we get the desired result (7.1).

Theorem 7.2. Let the function $f(z)$ defined by (1.4) be in the class $\mathcal{M}_p^{m, \lambda, \mu, \eta, \delta}(\beta, \alpha)$. Then $f(z)$ is p -valently starlike of order σ ($0 \leq \sigma < p$) in the disk $|z| < r_4$, where

$$r_4 = \inf_{n \in \mathbb{N}} \left\{ \frac{(p - \sigma) \left(\frac{p + \delta n}{p} \right)^m [n + p(1 - \beta\alpha)] \gamma_n(\lambda, \mu, \eta, p)}{p(1 - \alpha)(p + n - \sigma)} \right\}^{1/n} \quad (7.6)$$

and $\gamma_n(\lambda, \mu, \eta, p)$ given by (1.10). The result is sharp with the external function $f(z)$ given by (2.4).

Proof. It suffices to show that



$$\left| \frac{z f'(z)}{f(z)} - p \right| \leq p - \sigma, (|z| < r_4) \quad (7.7)$$

Indeed we have

$$\left| \frac{z f'(z)}{f(z)} - p \right| = \left| \frac{-\sum_{n=1}^{\infty} n a_{p+n} z^n}{1 - \sum_{n=1}^{\infty} a_{p+n} z^n} \right| \leq \frac{\sum_{n=1}^{\infty} n a_{p+n} |z|^n}{1 - \sum_{n=1}^{\infty} a_{p+n} |z|^n} \quad (7.8)$$

Hence (7.7) is true if

$$\sum_{n=1}^{\infty} n a_{p+n} |z|^n \leq (p - \sigma) - \sum_{n=1}^{\infty} (p - \sigma) a_{p+n} |z|^n, \quad (7.9)$$

that is, if

$$\sum_{n=1}^{\infty} \frac{(p + n - \sigma)}{(p - \sigma)} a_{p+n} |z|^n \leq 1 \quad (7.10)$$

By Theorem 2.1, (7.10) is true if

$$\frac{(p + n - \sigma)}{(p - \sigma)} |z|^n \leq \frac{\left(\frac{p + \delta n}{p}\right)^m [n + p(1 - \beta\alpha)] \gamma_n(\lambda, \mu, \eta, p)}{p(1 - \alpha)}, \quad (n \in \mathbb{N}). \quad (7.11)$$

Solving (7.11) for $|z|$, we get the desired result (7.6).

Theorem 7.3. Let the function $f(z)$ defined by (1.4) be in the class $\mathcal{M}_p^{m, \lambda, \mu, \eta, \delta}(\beta, \alpha)$. Then $f(z)$ is p -valently convex of order σ ($0 \leq \sigma < p$) in the disk $|z| < r_5$, where

$$r_5 = \inf_{n \in \mathbb{N}} \left\{ \frac{\left((p - \sigma) \left(\frac{p + \delta n}{p}\right)^m [n + p(1 - \beta\alpha)] \gamma_n(\lambda, \mu, \eta, p) \right)^{1/n}}{(p + n)(1 - \alpha)(p + n - \sigma)} \right\} \quad (7.12)$$

and $\gamma_n(\lambda, \mu, \eta, p)$ given by (1.10). The result is sharp with the external function $f(z)$ given by (2.4).

Proof. It suffices to show that

$$\left| 1 + \frac{z f''(z)}{f'(z)} - p \right| \leq p - \sigma, (|z| < r_5) \quad (7.13)$$

Indeed we have

$$\left| 1 + \frac{z f''(z)}{f'(z)} - p \right| = \left| \frac{-\sum_{n=1}^{\infty} n(p + n) a_{p+n} z^n}{p - \sum_{n=1}^{\infty} (p + n) a_{p+n} z^n} \right| \leq \frac{\sum_{n=1}^{\infty} n(p + n) a_{p+n} |z|^n}{p - \sum_{n=1}^{\infty} (p + n) a_{p+n} |z|^n} \quad (7.14)$$

Hence (7.13) is true if



$$\sum_{n=1}^{\infty} n(p+n) a_{p+n} |z|^n \leq p(p-\sigma) - \sum_{n=1}^{\infty} (p-\sigma)(p+n) a_{p+n} |z|^n \quad (7.15)$$

or

$$\sum_{n=1}^{\infty} \frac{(p+n)(n+p-\sigma)}{p(p-\sigma)} a_{p+n} |z|^n \leq 1 \quad (7.16)$$

By Theorem 2.1, (7.16) is true if

$$\frac{(p+n)(n+p-\sigma)}{(p-\sigma)} |z|^n \leq \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)] \gamma_n(\lambda, \mu, \eta, p)}{(1-\alpha)}, \quad (n \in \mathbb{N}). \quad (7.17)$$

Solving (7.17) for $|z|$, we get the desired result (7.12).

8- Class-Preserving Integral Operators

We prove that the integral operator $J_{c,p}$ defined by (1.14) is indeed a class-preserving operator for the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$.

Theorem 8.1. Let the function $f(z)$ defined by (1.4) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$. Also let $c > -p$. Then the function $F(z)$ defined by (1.14) is also in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$.

Proof . from (1.14) and (1.4), it easily seen that

$$F(z) = z^p - \sum_{n=1}^{\infty} A_{p+n} z^{p+n}, \quad (8.1)$$

where

$$A_{p+n} = \left(\frac{c+p}{c+p+n}\right) a_{p+n}, \quad (n \in \mathbb{N}; c > -p). \quad (8.2)$$

Since $c > -p$, we have

$$0 \leq A_{p+n} < a_{p+n}, \quad (n \in \mathbb{N}), \quad (8.3)$$

which, in view of Theorem 2.1, immediately yields Theorem 8.1.

Remark 2. Letting $c = 1 - p$ in Theorem 8.1, we obtain the following result.

Corollary 8.2. Let the function $f(z)$ defined by (1.4) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$. Then

$$G(z) = z^{p-1} \int_0^z \frac{f(t)}{t^p} dt \quad (8.4)$$

is also in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$.



Theorem 8.3. Let $c > -p$. Also let $F(z)$ be in class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$. Then the function $f(z)$ given by (1.14) is p -valent in the disk $|z| < r_6$ where

$$r_6 = \inf_{n \in \mathbb{N}} \left\{ \frac{\left(\left(\frac{p + \delta n}{p} \right)^m (c + p) [n + p(1 - \beta\alpha)] \gamma_n(\lambda, \mu, \eta, p) \right)^{1/n}}{(p + n)(c + p + n)(1 - \alpha)} \right\} \quad (8.5)$$

Proof. Assuming that

$$F(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n}, \quad (b_{p+n} \geq 0, p \in \mathbb{N}), \quad (8.6)$$

from (1.14), we get

$$f(z) = \frac{z^{1-c}}{c+p} \frac{d}{dz} (z^c F(z)) = z^p - \sum_{n=1}^{\infty} \left(\frac{c+p+n}{c+p} \right) b_{p+n} z^{p+n}, \quad (c > -p). \quad (8.7)$$

In order to prove the result, it suffices to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p, \quad (|z| < r_6). \quad (8.8)$$

Indeed we have

$$\begin{aligned} \left| \frac{f'(z)}{z^{p-1}} - p \right| &= \left| - \sum_{n=1}^{\infty} (p+n) \left(\frac{c+p+n}{c+p} \right) b_{p+n} z^n \right| \\ &\leq \sum_{n=1}^{\infty} (p+n) \left(\frac{c+p+n}{c+p} \right) b_{p+n} |z|^n, \end{aligned}$$

which yields the desired inequality in (8.8), provided that

$$\sum_{n=1}^{\infty} \frac{(p+n)(c+p+n)}{p(c+p)} b_{p+n} |z|^n \leq 1. \quad (8.9)$$

But, since the function $F(z)$ defined by (8.6) is in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta, \alpha)$, Theorem 2.1 gives us

$$\sum_{n=1}^{\infty} \frac{\left(\frac{p + \delta n}{p} \right)^m [n + p(1 - \beta\alpha)] \gamma_n(\lambda, \mu, \eta, p)}{p(1 - \alpha)} b_{p+n} \leq 1. \quad (8.10)$$

Thus the inequality (8.9), and hence also the inequality (8.8), will hold true if

$$\frac{(p+n)(c+p+n)}{p(c+p)} |z|^n \leq \frac{\left(\frac{p + \delta n}{p} \right)^m [n + p(1 - \beta\alpha)] \gamma_n(\lambda, \mu, \eta, p)}{p(1 - \alpha)} \quad (n \in \mathbb{N}),$$



that is, if

$$|z| \leq \left\{ \frac{\left(\frac{p+\delta n}{p}\right)^m (c+p)[n+p(1-\beta\alpha)]\gamma_n(\lambda, \mu, \eta, p)}{(p+n)(c+p+n)(1-\alpha)} \right\}^{1/n} \quad (n \in \mathbb{N}),$$

which leads us precisely to the main assertion of Theorem 8.3.

9- Integral Means Inequalities

Applying Lemma 1.3, we prove the following theorem.

Theorem 9.1. Let $\tau > 0$; $\lambda \geq 0$; $\mu < p + 1$; $\eta \geq \lambda \left(1 - \frac{p+2}{\mu}\right)$; $\delta \geq 0$; $m \in \mathbb{N}_0$; $0 < \beta \leq 1$; $0 \leq \alpha < 1$; $\lambda, \mu, \eta \in \mathbb{R}$ and $p \in \mathbb{N}$. If $f(z) \in \mathcal{M}_p^{m, \lambda, \mu, \eta, \delta}(\beta, \alpha)$, then for $z = re^{i\theta}$ and $0 < r < 1$, we have

$$\int_0^{2\pi} |f(re^{i\theta})|^\tau d\theta \leq \int_0^{2\pi} |f_{p+1}(re^{i\theta})|^\tau d\theta \quad (9.1)$$

where

$$f_{p+1}(z) = z^p - \frac{p(1-\alpha)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)] \gamma_1(\lambda, \mu, \eta, p)} z^{p+1} \quad (9.2)$$

and $\gamma_n(\lambda, \mu, \eta, p)$ given by (1.10).

Proof. Let $f(z)$ of the form (1.4) and $f_{p+1}(z)$ of the form (9.2), then we must show that

$$\int_0^{2\pi} \left| 1 - \sum_{n=1}^{\infty} a_{p+n} z^n \right|^\tau d\theta \leq \int_0^{2\pi} \left| 1 - \frac{p(1-\alpha)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)] \gamma_1(\lambda, \mu, \eta, p)} z \right|^\tau d\theta.$$

By Lemma 1.3, it suffices to show that

$$1 - \sum_{n=1}^{\infty} a_{p+n} z^n < 1 - \frac{p(1-\alpha)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)] \gamma_1(\lambda, \mu, \eta, p)} z$$

Setting

$$1 - \sum_{n=1}^{\infty} a_{p+n} z^n = 1 - \frac{p(1-\alpha)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)] \gamma_1(\lambda, \mu, \eta, p)} w(z). \quad (9.3)$$

From (9.3) and (2.1), we obtain



$$|w(z)| = \left| \sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)] \gamma_1(\lambda, \mu, \eta, p)}{p(1-\alpha)} a_{p+n} z^n \right|$$
$$\leq |z| \sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)] \gamma_n(\lambda, \mu, \eta, p)}{p(1-\alpha)} a_{p+n} \leq |z| < 1.$$

This completes the proof of the Theorem 9.1.

Remark 3. Letting $p = 1, \lambda = \mu = \delta = 0$ and $\beta = 1$, in Theorem 9.1, we get the integral means inequality for the class $T^*(\alpha)$.

Corollary 9.2. Let $\tau > 0$. If $f(z) \in T^*(\alpha)$, then for $z = re^{i\theta}$ and $0 < r < 1$, we have

$$\int_0^{2\pi} |f(re^{i\theta})|^\tau d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^\tau d\theta \quad (9.4)$$

where

$$f_2(z) = z - \frac{1-\alpha}{2-\alpha} z^2. \quad (9.5)$$

Remark 4. If we take $\alpha = 0$ in $T^*(\alpha)$ of Corollary 9.2, we obtain the result proved by Silverman [9].

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الفهرس

الصفحة	اسم الباحث	عنوان البحث	ر.ت
25-3	زهرة المهدي أبوراس فاطمة أحمد قناو	التسرب الدراسي لدي طلاب الجامعات	1
43-26	علي فرج حامد فاطمة جبريل القايد	استعمالات الأرض الزراعية في منطقة سوق الخميس	2
57-44	ابتسام عبد السلام كشيبي	تأثير صناعة الإسمنت على البيئة مصنع إسمنت لبدة نموذجاً دراسة في الجغرافية الصناعي	3
84-58	عطية صالح علي الربيعي خالد رمضان الجربوع منصور علي سالم خليفة	مفهوم الشعر عند نقاد القرن الرابع الهجري	4
106-85	فتحية علي جعفر أمنة محمد العكاشي ربيعة عثمان عبد الجليل	جودة الحياة لدى طلبة كلية التربية بالخميس	5
128-107	Ebtisam Ali Haribash A.A.H. Abd EL-Mwla	An Active-Set Line-Search Algorithm for Solving Multi-Objective Transportation Problem	6
140-129	مفتاح سالم ثبوت	آليات بناء النص عند بدر شاكر السياب قراءة في قصيدة تموز جيكور	7
155-141	مفتاح ميلاد الهديف جمعة عبد الحميد شنيب	الجرائم الالكترونية	8
176-156	Suad H. Abu-Janah	On the fine spectrum of the generalized difference operator over the Hahn sequence space $B(r, s)_h$	9
201-177	فوزية محمد الحوات سالمة محمد ضو	دراسة تأثير التضاد الكيميائي Allelopathy لمستخلصات بعض النباتات الطبية على نسبة الانبات ونمو نبات القمح Triticum aestivum L.	10
219-202	سليمة محمد خضر	الأعداد الضبابية	11
240-220	S. M. Amsheri N. A. Aboutfeerah	On a certain class of P -valent functions with negative coefficients	12
241-253	Abdul Hamid Alashhab	L'écriture de la violence dans la littérature africaine et plus précisément dans le théâtre Ivoirien Mhoi-Ceul comédie en 5 tableaux de Bernard B. Dadié	13
254-265	Shibani K. A. Zaggout F. N	Electronic Specific Heat of Multi Levels Superconductors Based on the BCS Theory	14



266-301	خالد رمضان محمد الجربوع عطية صالح علي الربيعي	أعراض الشعر المستجدة في العصر العباسي	15
302-314	M. J. Saad, N. Kumaresan Kuru Ratnavelu	Oscillation Criterion for Second Order Nonlinear Differential Equations	16
315-336	صالح عبد السلام الكيلاني ساره مفتاح الزني فدوى خليل سالم	القيم الجمالية لفن الفسيفساء عند العرب	17
337-358	عبدالمعظم امحمد سالم	مفهوم السلطة عند المعتزلة وإخوان الصفاء	18
359-377	أسماء حامد عبدالحفيظ اعليجه	مستوى الوعي البيئي ودور بعض القيم الاجتماعية في رفعه لدى عينة من طلاب كلية الآداب الواقعة داخل نطاق مدينة الخمس.	19
378-399	بنور ميلاد عمر العماري	المؤسسات التعليمية ودورها في الوقاية من الانحراف والجريمة	20
400-405	Mohammed Ebraheem Attaweel Abdulah Matug Lahwal	Application of Sawi Transform for Solving Systems of Volterra Integral Equations and Systems of Volterra Integro-differential Equations	21
406-434	Eman Fathullah Abusteen	The perspectives of Second Year Students At Faculty of Education in EL-Mergib University towards Implementing of Communicative Approach to overcome the Most Common Challenges In Learning Speaking Skill	22
435-446	Huda Aldweby Amal El-Aloul	Sufficient Conditions of Bounded Radius Rotations for Two Integral Operators Defined by q-Analogue of Ruscheweyh Operator	23
447-485	سعاد مفتاح أحمد مرجان	مستوى الوعي بمخاطر التلوث البيئي لدى معلمي المرحلة الثانوية بمدينة الخمس	24
486-494	Hisham Zawam Rashdi Mohammed E. Attaweel	A New Application of Sawi Transform for Solving Ordinary differential equations with Variable Coefficients	25
495-500	محمد على أبو النور فرج مصطفى الهدار بشير على الطيب	استخدام التحليل الإحصائي لدراسة العلاقة بين أنظمة الري وكمية المياه المستهلكة بمنطقة سوق الخميس - الخمس	26
501-511	نرجس ابراهيم محمد شنيب	التقييم المنهجي للمواد الرياضية و الاحصائية نسبة الى المواد التخصصية لعلوم الحاسوب	27
512-536	بشري محمد الهيلي حنان سعيد العوراني عفاف محمد بالحاج	طرق التربية الحديثة للأطفال	28
537-548	ضو محمد عبد الهادي فاروق مصطفى ايور اوي زهرة صبحي سعيد نجاح عمران المهدي	دراسة للحد من التلوث الكهرومغناطيسي باستخدام مركب ثاني أكسيد الحديد مع بوليمر حمض الاكتيك	29



549-563	Ali ahmed baraka Abobaker m albaboh Abdussalam a alashhab	Cloud Computing Prototype for Libya Higher Education Institutions: Concept, Benefits and Challenges	30
564-568	Muftah B. Eldeeb	Euphemism in Arabic Language: The case with Death Expressions	31
569-584	Omar Ismail Elhasadi Mohammed Saleh Alsayd Elhadi A. A. Maree	Conjugate Newton's Method for a Polynomial of degree m+1	32
585-608	آمنة سالم عبد القادر قدرو آلاء عبدالسلام محمد سويسي ليلى علي محمد الجاعوك	الصحة النفسية وعلاقتها بتقدير الذات لدى عينة من طلبة كلية الآداب والعلوم / مسلاته	33
609-625	نجاه سالم عبد الله زريق	المساندة الاجتماعية لدى عينة من المعلمات بمدينة قصر الأخيار وعلاقتها ببعض المتغيرات الديموغرافية "دراسة ميدانية"	34
626-640	محمد سالم ميلاد العابر	"أي" بين الاسمية والفعلية عاملة ومعمولة	35
641-659	إبراهيم فرج الحويج	التمييز في القرآن الكريم سورة الكهف أنموذجا	36
660-682	عبد السلام ميلاد المركز رجعة سعيد الجنقاوي	الموارد الطبيعية و البشرية السياحية بمدينة طرابلس (بليبيا)	37
683-693	Ibrahim A. Saleh Abdelnaser S. Saleh Youssif S M Elzawiei Farg Gait Boukhrais	Influence of Hydrogen content on structural and optical properties of doped nano-a-Si:H/a-Ge: H multilayers used in solar cells	38
694-720	فرج رمضان مفتاح الشبيلي	أجوبة الشيخ علي بن أبي بكر الحضيري (ت: 1061 هـ - 1650 م)	39
721-736	علي خليفة محمد أجولي	مفهوم الهوية عند محمد أركون	40
737-742	Mahmoud Ahmed Shaktour	Current –mode Kerwin, Huelsman and Newcomb (KHN) By using CDTA	41
743-772	Salem Msauad Adrugi Tareg Abdusalam Elawaj Milad Mohamed Alhwat	University Students' Attitudes towards Blended Learning in Libya: Empirical Study	42
773-783	Alhusein M. Ezarzah Aisha S. M. Amer Adel D. El werfalyi Khalil Salem Abulsba Mufidah Alarabi Zagloom	Integrated Protected Areas	43
784-793	عبد الرحمن المهدي ابومنجل	المظاهرات بين المانعين والمجوزين	44
794-817	رضا القذافي بشير الاسمر	ترجيحات الامام الباجي من خلال كتابه المنتقى " من باب العنافة والولاء الى كتاب الجامع "	45



مجلة التربوي
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معامل التأثير العربي 1.5
العدد 20

818-829	Fadela M. Elzalet Sami A. S. Noba omar M. A. kaboukah	IDENTIFICATION THE OPTIMUM PRODUCTION PROCESS OF THE HYDROGEN GAS	46
830	الفهرس		