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معامل التأثير العربي 1.5
العدد 20

مجلة التربوي
مجلة علمية محكمة تصدر عن كلية التربية
جامعة المرقب

العدد العشرون
يناير 2022م

هيئـــة تحريـر
مجلة التربوي

- المجلة ترحب بما يرد عليها من أبحاث وعلى استعداد لنشرها بعد التحكيم .
- المجلة تحترم كل الاحترام آراء المحكمين وتعمل بمقتضاها .
- كافة الآراء والأفكار المنشورة تعبر عن آراء أصحابها ولا تتحمل المجلة تبعاتها .
- يتحمل الباحث مسؤولية الأمانة العلمية وهو المسؤول عما ينشر له .
- البحوث المقدمة للنشر لا ترد لأصحابها نشرت أو لم تنشر .

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ضوابط النشر:

يشترط في البحوث العلمية المقدمة للنشر أن يراعى فيها ما يأتي :

- أصول البحث العلمي وقواعده .
- ألا تكون المادة العلمية قد سبق نشرها أو كانت جزءا من رسالة علمية .
- يرفق بالبحث تزكية لغوية وفق أنموذج معد .
- تعدل البحوث المقبولة وتصحح وفق ما يراه المحكمون .
- التزام الباحث بالضوابط التي وضعتها المجلة من عدد الصفحات ، ونوع الخط ورقمه ، والفترات الزمنية الممنوحة للتعديل ، وما يستجد من ضوابط تضعها المجلة مستقبلا .

تنبيهات :

- للمجلة الحق في تعديل البحث أو طلب تعديله أو رفضه .
- يخضع البحث في النشر لأولويات المجلة وسياستها .
- البحوث المنشورة تعبر عن وجهة نظر أصحابها ، ولا تعبر عن وجهة نظر المجلة .

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On a certain class of p-valent functions with negative coefficients

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Abstract

In this paper we introduce a new class of p-valent starlike functions with negative coefficients defined by fractional derivative operator. We obtain coefficient bounds, distortion inequalities, Hadamard product, linear combinations and inclusion theorems. Also, we find extreme points and radii of close-to-convexity, starlikeness and convexity for this class. The integral preserving properties and integral means inequalities are also determined.

Keywords: multivalent (p-valent) functions, starlike functions, convex functions, close-to-convex functions, fractional derivatives, Hadamard product, integral means.

1- Introduction and Definitions

Let $A(p)$ denote the class of functions defined by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}\, z^{p+n} \;, \qquad (p \in \mathbb{N}). \tag{1.1}$$

which are analytic and multivalent (or p-valent) in the open unit disk $\mathcal{U} = \{z: |z| < 1\}$. We write $A(1) = A$. If f and g are analytic in $\mathcal{U}$, we say that f is subordinate to g, written symbolically as $f \prec g$ or $f(z) \prec g(z), z \in \mathcal{U}$, if there exists a Schwarz function $w(z)$ which is analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g\big(w(z)\big), z \in \mathcal{U}$.

A function $f(z) \in A(p)$ is said to be p-valent starlike of order α if $f(z)$ satisfies the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \;, \qquad (0 \le \alpha < p\,; p \in \mathbb{N}; z \in \mathcal{U}) \tag{1.2}$$

We denote by $S^*(p,\alpha)$ the class of all such functions. A function $f(z) \in A(p)$ is said to be p-valent convex of order α if $f(z)$ satisfies the condition

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha, \qquad (0\leq\alpha<p\,;p\in\mathbb{N};z\in\mathcal{U}) \tag{1.3}$$

Let $K(p,\alpha)$ denote the class of all those functions which are p-valent convex of order α. The class $S^*(p,\alpha)$ was introduced by Patil and Thakare [6], and the class $K(p,\alpha)$ was introduced by Owa [5].

Let $T(p)$ denote the subclass of $A(p)$ consisting of functions of the form

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}\,z^{p+n} \qquad \left(a_{p+n}\geq 0\,,p\in\mathbb{N},z\in\mathcal{U}\right). \tag{1.4}$$

We denote by $T^*(p,\alpha)$ and $C(p,\alpha)$ the classes obtained by taking intersections, respectively, of the classes $S^*(p,\alpha)$ and $K(p,\alpha)$ with the class $T(p)$. The classes $T^*(p,\alpha)$ and $C(p,\alpha)$ were introduced by Owa [5]. In particular, the classes $T^*(1,\alpha)=T^*(\alpha)$ and $C(1,\alpha)=C(\alpha)$ when $p=1$ were studied by Silverman [8].

Let the functions $f_i(z),(i=1,2)$ be defined by

$$f_i(z)=z^p-\sum_{n=1}^{\infty}a_{p+n,i}\ z^{p+n}\ , \qquad \left(a_{p+n,i}\geq 0;\ p\in\mathbb{N}\right). \tag{1.5}$$

The Hadmard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1*f_2)(z)=z^p-\sum_{n=1}^{\infty}a_{p+n,1}\ a_{p+n,2}\ z^{p+n} \tag{1.6}$$

Definition 1.1.[1,2,10]. Let $0\leq\lambda<1$, and $\mu,\eta\in\mathbb{R}$, the fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}$ is defined in terms of Gauss's hypergeometric function ${}_2F_1$ as follows

$$J_{0,z}^{\lambda,\mu,\eta}f(z)=\frac{d}{dz}\left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)}\int_0^z(z-\xi)^{-\lambda}f(\xi)\ {}_2F_1\left(\mu-\lambda,1-\eta;1-\lambda;1-\frac{\xi}{Z}\right)d\xi\right) \tag{1.7}$$

where $f(z)$ is analytic function in a simply- connected region of the z-plane containing the origin with the order $f(z)=O(|z|^{\varepsilon})$, $z\to 0$, where $\varepsilon>\max\{0,\mu-\eta\}-1$, and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi>0$.

We note that, $J_{0,z}^{\lambda,\lambda,\eta}f(z)=D_z^{\lambda}f(z), 0\leq\lambda<1$ is the fractional derivative operator considered by Owa [4].

The fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta,p}f(z)$ of a function $f(z)$ in $A(p)$ is defined by

$$M_{0,z}^{\lambda,\mu,\eta,p}f(z)=\frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\eta)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)}z^{\mu}J_{0,z}^{\lambda,\mu,\eta}f(z) \qquad (1.8)$$

$$(\lambda\geq 0;\ \mu<p+1;\ \eta>\max(\lambda,\mu)-p-1; p\in\mathbb{N})$$

The operator $M_{0,z}^{\lambda,\mu,\eta,p}f(z)$ was studied by Amsheri and Zharkova [1]. (see also [10]).

Recently, Zayed et al. [10] introduced the operator $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)$ of a function $f(z)$ in $A(p)$ for $m\in\mathbb{N}_o=\mathbb{N}\cup\{0\}$ and $\delta\geq 0$ and defined by:

$$N_{0,z}^{0,\lambda,\mu,\eta,\delta,p}f(z)=M_{0,z}^{\lambda,\mu,\eta,p}f(z)$$

$$N_{0,z}^{1,\lambda,\mu,\eta,\delta,p}f(z)=N_{0,z}^{\lambda,\mu,\eta,\delta,p}f(z)$$

$$=(1-\delta)M_{0,z}^{\lambda,\mu,\eta,p}f(z)+\delta\frac{z}{p}\left(M_{0,z}^{\lambda,\mu,\eta,p}f(z)\right)'$$

and (in general),

$$N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)=N_{0,z}^{\lambda,\mu,\eta,\delta,p}\left(N_{0,z}^{m-1,\lambda,\mu,\eta,\delta,p}f(z)\right)$$

$$=z^p+\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^m\gamma_n(\lambda,\mu,\eta,p)a_{p+n}z^{p+n} \qquad (1.9)$$

where

$$\gamma_n(\lambda,\mu,\eta,p)=\frac{(p+1)_n(p+1-\mu+\eta)_n}{(p+1-\mu)_n(p+1-\lambda+\eta)_n}\ , \qquad (n\in\mathbb{N}) \qquad (1.10)$$

Motivated essentially by aforementioned works, we introduce a new class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ of analytic and p-valent functions $f(z)$ belonging to the class $T(p)$ by using the operator $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)$ as follows:

Definition 1.2. The function $f(z)\in T(p)$ is said to be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ if it satisfies

$$\frac{\frac{1}{p}z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)'}{(1-\beta)z^p+\beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}\prec\frac{1+(1-2\alpha)z}{1-z}\ , \qquad (z\in\mathcal{U};\ p\in\mathbb{N}) \qquad (1.11)$$

For $m\in\mathbb{N}_0$; $\lambda\geq 0$; $\mu<p+1$; $\eta>\max(\lambda,\mu)-p-1; 0<\beta\leq 1; 0\leq\alpha<1; \delta\geq 0$. The condition (1.11) is equivalent to

$$\left|\frac{\dfrac{\frac{1}{p}z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)'}{(1-\beta)z^p+\beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}-1}{\dfrac{\frac{1}{p}z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)'}{(1-\beta)z^p+\beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}+1-2\alpha}\right|<1 \qquad (z\in\mathcal{U};\ p\in\mathbb{N}) \tag{1.12}$$

Easily we can deduce that,

$$N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)=z^p-\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^m\gamma_n(\lambda,\mu,\eta,p)a_{p+n}z^{p+n} \quad \left(a_{p+n}\geq 0, p\in\mathbb{N}\right) \tag{1.13}$$

where $\gamma_n(\lambda,\mu,\eta,p)$ is given by (1.10). We observe that,

1- For $\lambda=\mu=\delta=0$, the class $\mathcal{M}_p^{,m,0,0,\eta,0}(\beta,\alpha)=\mathcal{M}_p(\beta,\alpha)$ (Selvaraj et al. [7]).
2- For $\lambda=\mu=\delta=0$ and $\beta=1$, the class $\mathcal{M}_p^{,m,0,0,\eta,0}(1,\alpha)=T^*(p,\alpha)$ (Owa [5]).
3- For $\lambda=\mu=\delta=0$ and $p=\beta=1$, the class $\mathcal{M}_1^{,m,0,0,\eta,0}(1,\alpha)=T^*(\alpha)$ (Silverman [8]).

In the present paper, we obtain coefficient bounds, distortion inequalities, Hadamard product, linear combinations and inclusion theorems for the functions belonging to the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Also, we find extreme points and radii of close-to-convexity, starlikeness and convexity for this class. Finally, we determine the integral means inequalities and a class-preserving integral operator of the form

$$F(z)=\left(J_{c,p}f\right)(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)\,dt, \qquad (c>-p) \tag{1.14}$$

In order to prove our results in section 9 we shall need the following lemma.

Lemma 1.3. [3]. If the functions $f(z)$ and $g(z)$ are analytic in $\mathcal{U}$ with $g(z)\prec f(z)$, then

$$\int_0^{2\pi}\left|f(re^{i\theta})\right|^\tau d\theta \leq \int_0^{2\pi}\left|g(re^{i\theta})\right|^\tau d\theta\,, \quad \left(z=re^{i\theta}, 0<r<1\right) \tag{1.15}$$

2- Coefficient Bounds

Theorem 2.1. Let the function $f(z)$ be defined by (1.4). Then $f(z)$ belongs to the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ if and only if

$$\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^m[n+p(1-\beta\alpha)]\,\gamma_n(\lambda,\mu,\eta,p)a_{p+n}\leq p(1-\alpha) \tag{2.1}$$

where $\gamma_n(\lambda,\mu,\eta,p)$ is given by (1.10).

Proof. Since $f(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$, then

$$\left| \frac{\frac{1}{p} z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)\right)' - \left((1-\beta)z^p + \beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)\right)}{\frac{1}{p} z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)\right)' + (1-2\alpha)\left((1-\beta)z^p + \beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)\right)} \right| < 1 . \qquad (2.2)$$

It follows from (2.2) that

$$\operatorname{Re}\left\{ \frac{\sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)\left[\left(\frac{p+n}{p}\right) - \beta\right] a_{p+n} z^n}{2(1-\alpha) - \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)\left[\left(\frac{p+n}{p}\right) + \beta(1-2\alpha)\right] a_{p+n} z^n} \right\} < 1 .$$

Choosing values of z on the real axis so that $\frac{\frac{1}{p} z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)\right)'}{(1-\beta)z^p + \beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)}$ is real, and letting $z \to 1^-$ through real axis, we have

$$\sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)\left[\left(\frac{p+n}{p}\right) - \beta\right] a_{p+n}$$

$$\le 2(1-\alpha) - \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)\left[\left(\frac{p+n}{p}\right) + \beta(1-2\alpha)\right] a_{p+n}$$

which gives the desired assertion (2.1). Conversely, let the inequality (2.1) holds true and let $|z| = 1$. Then we have

$$\left|\frac{1}{p} z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)\right)' - \left((1-\beta)z^p + \beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)\right)\right|$$

$$- \left|\frac{1}{p} z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)\right)' + (1-2\alpha)\left((1-\beta)z^p + \beta N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)\right)\right|$$

$$= \left| - \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)\left[\left(\frac{p+n}{p}\right) - \beta\right] a_{p+n} z^{p+n} \right|$$

$$- \left| 2(1-\alpha)z^p - \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)\left[\left(\frac{p+n}{p}\right) + \beta(1-2\alpha)\right] a_{p+n} z^{p+n} \right|$$

$$\le 2 \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m \left[\left(\frac{p+n}{p}\right) - \alpha\beta\right] \gamma_n(\lambda,\mu,\eta,p) a_{p+n} - 2(1-\alpha) \le 0.$$

Hence by the maximum modulus theorem, $f(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. This completes the proof.

Corollary 2.2. Let the function $f(z)$ be defined by (1.4) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$, then

$$a_{p+n} \leq \frac{p(1-\alpha)}{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\,\gamma_n(\lambda,\mu,\eta,p)} \quad , \qquad (p,n \in \mathbb{N}) \qquad (2.3)$$

where $\gamma_n(\lambda,\mu,\eta,p)$ is given by (1.10). The result (2.3) is sharp for a function of the form:

$$f(z) = z^p - \frac{p(1-\alpha)}{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\,\gamma_n(\lambda,\mu,\eta,p)} z^{p+n} \quad , \quad (p,n \in \mathbb{N}) \qquad (2.4)$$

Remark 1. Letting $p = 1, \lambda = \mu = \delta = 0$ and $\beta = 1$ in Theorem 2.1 and Corollary 2.2 respectively, we obtain the results were proved by Silverman [8].

3- Distortion Inequalities

Theorem 3.1. Let $\lambda,\mu,\eta \in \mathbb{R}$, such that

$$\lambda \geq 0, \mu < p+1, \eta \geq \lambda\left(1-\frac{p+2}{\mu}\right), \delta \geq 0, m \in \mathbb{N}_0, 0 < \beta \leq 1, 0 \leq \alpha < 1 \text{ and } p \in \mathbb{N}. \qquad (3.1)$$

Let the function $f(z)$ defined by (1.4) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Then

$$|f(z)| \geq |z|^p - \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)} |z|^{p+1}, \qquad (3.2)$$

$$|f(z)| \leq |z|^p + \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)} |z|^{p+1}, \qquad (3.3)$$

$$|f'(z)| \geq p|z|^{p-1} - \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1-\mu+\eta)} |z|^p, \qquad (3.4)$$

and

$$|f'(z)| \leq p|z|^{p-1} + \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1-\mu+\eta)} |z|^p, \qquad (3.5)$$

for $z \in \mathcal{U}$. The estimates for $|f(z)|$ and $|f'(z)|$ are sharp.

Proof. We observe that the function $\gamma_n(\lambda,\mu,\eta,p)$ defined by (1.10) satisfy the inequality $\gamma_n(\lambda,\mu,\eta,p) \leq \gamma_{n+1}(\lambda,\mu,\eta,p)$, $\forall n \in \mathbb{N}$, provided that $\eta \geq \lambda\left(1-\frac{p+2}{\mu}\right)$.

Thereby, showing that $\gamma_n(\lambda,\mu,\eta,p)$ is non-decreasing. Thus under the conditions stated in (3.1), we have

$$0<\frac{(p+1)(p+1-\mu+\eta)}{(p+1-\mu)(p+1-\lambda+\eta)}=\gamma_1(\lambda,\mu,\eta,p)\le\gamma_n(\lambda,\mu,\eta,p)\quad \forall n\in\mathbb{N}\ , \tag{3.6}$$

for $f(z)\in\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$, in view of Theorem 2.1, we have

$$\frac{\left(\frac{p+\delta}{p}\right)^m[1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)}{(p+1-\mu)(p+1-\lambda+\eta)}\sum_{n=1}^{\infty}a_{p+n}$$
$$\le\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^m[n+p(1-\beta\alpha)]\,\gamma_n(\lambda,\mu,\eta,p)a_{p+n}\le p(1-\alpha)\ , \tag{3.7}$$

which gives

$$\sum_{n=1}^{\infty}a_{p+n}\le\frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m[1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)} \tag{3.8}$$

Consequently, we obtain

$$|f(z)|\ge|z|^p-|z|^{p+1}\sum_{n=1}^{\infty}a_{p+n}$$
$$\ge|z|^p-\frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m[1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)}|z|^{p+1} \tag{3.9}$$

and

$$|f(z)|\le|z|^p+|z|^{p+1}\sum_{n=1}^{\infty}a_{p+n}$$
$$\le|z|^p+\frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m[1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)}|z|^{p+1} \tag{3.10}$$

which prove the assertions (3.2) and (3.3) of Theorem 3.1. Furthermore, from Theorem 2.1, we note that

$$\sum_{n=1}^{\infty}(p+n)\,a_{p+n}\le\frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m[1+p(1-\beta\alpha)](p+1-\mu+\eta)}\,. \tag{3.11}$$

Thus, we have

$$\left|f^{'}(z)\right| \geq p|z|^{p-1} - |z|^p \sum_{n=1}^{\infty}(p+n)a_{p+n} \quad (3.12)$$

and

$$\left|f^{'}(z)\right| \leq p|z|^{p-1} + |z|^p \sum_{n=1}^{\infty}(p+n)a_{p+n}\,. \quad (3.13)$$

On using (3.12), (3.13) and (3.11), we arrive at the desired results (3.4) and (3.5).

Finally, we can prove that the estimates for $|f(z)|$ and $\left|f^{'}(z)\right|$ are sharp by taking the function

$$f(z) = z^p - \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)} z^{p+1} \quad (3.14)$$

Corollary 3.2. Let the function $f(z)$ be defined by (1.4) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Then $f(z)$ is included in a disk with centre at the origin and radius r_1 given by

$$r_1 = 1 + \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1)(p+1-\mu+\eta)} \quad (3.15)$$

$f'(z)$ is included in a disk with centre at the origin and radius r_2 given by

$$r_2 = p + \frac{p(1-\alpha)(p+1-\lambda+\eta)(p+1-\mu)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)](p+1-\mu+\eta)} \quad (3.16)$$

4- Hadamard Product

Theorem 4.1. Let $\lambda \geq 0;\ \mu < p+1;\ \eta \geq \lambda\left(1-\frac{p+2}{\mu}\right);\ \delta \geq 0;\ m \in \mathbb{N}_0;\ 0 < \beta \leq 1;\ 0 \leq \alpha < 1;\ \lambda,\mu,\eta \in \mathbb{R}$ and $p \in \mathbb{N}$, and let the functions $f_i(z)(i=1,2)$ defined by (1.5) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$.Then $(f_1 * f_2)(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\sigma)$ where

$$\sigma \leq \inf_{n\in\mathbb{N}}\left\{\frac{\nu(n) - p(p+n)(1-\alpha)^2}{\nu(n) - p^2\beta(1-\alpha)^2}\right\} \quad (4.1)$$

where

$$\nu(n) = \left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]^2 \gamma_n(\lambda,\mu,\eta,p) \quad (4.2)$$

Proof. It suffices to prove that

$$\sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\sigma)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\sigma)} a_{p+n,1}\ a_{p+n,2} \ \leq 1. \qquad (4.3)$$

Since

$$\sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} a_{p+n,1} \ \leq 1, \qquad (4.4)$$

and

$$\sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} a_{p+n,2} \ \leq 1\ . \qquad (4.5)$$

By the Cauchy – Schwarz inequality, we have

$$\sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \sqrt{a_{p+n,1}\ a_{p+n,2}} \ \leq 1 \qquad (4.6)$$

Thus, we need to find the largest σ such that

$$\sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\sigma)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\sigma)} a_{p+n,1}\ a_{p+n,2}$$

$$\leq \sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \sqrt{a_{p+n,1}\ a_{p+n,2}}\ , \qquad (4.7)$$

or, equivalently, that

$$\sqrt{a_{p+n,1}\ a_{p+n,2}} \leq \frac{[n+p(1-\beta\alpha)](1-\sigma)}{[n+p(1-\beta\sigma)](1-\alpha)}\ . \qquad (4.8)$$

In view of (4.6), it is sufficient to find the largest σ such that

$$\frac{p(1-\alpha)}{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)} \leq \frac{[n+p(1-\beta\alpha)](1-\sigma)}{[n+p(1-\beta\sigma)](1-\alpha)}, \quad (n\in\mathbb{N}). \qquad (4.9)$$

The inequality (4.9) yields

$$\sigma \leq \left\{\frac{\nu(n)-p(p+n)(1-\alpha)^2}{\nu(n)-p^2\beta(1-\alpha)^2}\right\}$$

where

$$v(n) = \left(\frac{p+\delta n}{p}\right)^m [n + p(1-\beta\alpha)]^2 \gamma_n(\lambda,\mu,\eta,p) \,,$$

and this the inequality gives the required result.

Corollary 4.2. For $f_i(z)(i = 1,2)$ as Theorem 4.1, we have

$$h(z) = z^p - \sum_{n=1}^{\infty} \sqrt{a_{p+n,1}\, a_{p+n,2}}\; z^{p+n}, \qquad (p \in \mathbb{N}) \tag{4.10}$$

belongs to the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$.

Proof. The result follows from the inequality (4.6).

Theorem 4.3. Let the function $f(z)$ defined by (1.4) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Also let

$$g(z) = z^p - \sum_{n=1}^{\infty} b_{p+n}\; z^{p+n} \;, \qquad \left(|b_{p+n}| \le 1, p \in \mathbb{N}\right) \tag{4.11}$$

Then $(f * g)(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$.

Proof. Since

$$\sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m [n + p(1-\beta\alpha)]\, \gamma_n(\lambda,\mu,\eta,p) |a_{p+n} b_{p+n}|$$

$$= \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m [n + p(1-\beta\alpha)]\, \gamma_n(\lambda,\mu,\eta,p) a_{p+n} |b_{p+n}|$$

$$\le \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m [n + p(1-\beta\alpha)]\, \gamma_n(\lambda,\mu,\eta,p) a_{p+n}$$

$$\le p(1-\alpha).$$

By Theorem 2.1, it follows that

$$(f * g)(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha).$$

5- Linear Combinations and Inclusion Theorems

We shall prove that the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ is closed under arithmetic mean and under convex linear combinations.

Theorem 5.1. Let the functions $f_i(z)(i = 1, 2, \dots\dots m)$ defined by (1.5) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Then the function

$$h(z) = z^p - \frac{1}{m}\sum_{n=1}^{\infty}\left(\sum_{i=1}^{m} a_{p+n,i}\right) z^{p+n} \tag{5.1}$$

is also in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$.

Proof. Since $f_i(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$, by using Theorem 2.1, we have

$$\sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} a_{p+n,i} \leq 1 \quad ,(i = 1,2,3,\dots,m) \tag{5.2}$$

so,

$$\sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \left(\frac{1}{m}\sum_{i=1}^{m} a_{p+n,i}\right)$$

$$= \frac{1}{m}\sum_{i=1}^{m}\sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} a_{p+n,i} \leq 1 \tag{5.3}$$

which shows that $h(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$, and the proof of Theorem 5.1 is completed.

Theorem 5.2. Let $\lambda \geq 0$; $\mu < p+1$; $\eta \geq \lambda\left(1-\frac{p+2}{\mu}\right)$; $\delta \geq 0$; $m \in \mathbb{N}_0$; $0 < \beta \leq 1$; $0 \leq \alpha < 1$; $\lambda,\mu,\eta \in \mathbb{R}$ and $p \in \mathbb{N}$, and let the functions $f_i(z)(i = 1,2)$ defined by (1.5) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Then the function

$$h(z) = z^p - \sum_{n=1}^{\infty}(a_{p+n,1}^2 + a_{p+n,2}^2)\, z^{p+n} \qquad ,(p \in \mathbb{N}) \tag{5.4}$$

belongs to the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\sigma)$,where

$$\sigma \leq \inf_{n\in\mathbb{N}}\left\{\frac{\nu(n) - 2p(p+n)(1-\alpha)^2}{\nu(n) - 2p^2\beta(1-\alpha)^2}\right\} \tag{5.5}$$

Where $\nu(n)$ given by (4.2).

Proof. By virtue of Theorem 2.1, we obtain

$$\sum_{n=1}^{\infty}\left\{\frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)}\right\}^2 a_{p+n,1}^2$$

$$\leq \left\{ \sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \; a_{p+n,1} \right\}^2 \leq 1 \qquad (5.6)$$

and

$$\sum_{n=1}^{\infty} \left\{ \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \right\}^2 a_{p+n,2}^2$$

$$\leq \left\{ \sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \; a_{p+n,2} \right\}^2 \leq 1. \qquad (5.7)$$

It follows from (5.6) and (5.7) that

$$\sum_{n=1}^{\infty} \frac{1}{2} \left\{ \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \right\}^2 (a_{p+n,1}^2 + a_{p+n,2}^2) \leq 1. \qquad (5.8)$$

Therefore, we need to find the largest σ such that

$$\sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\sigma)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\sigma)} (a_{p+n,1}^2 + a_{p+n,2}^2) \leq 1. \qquad (5.9)$$

Thus, it is sufficient to show that

$$\frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\sigma)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\sigma)}$$

$$\leq \frac{1}{2} \left\{ \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \right\}^2, \quad (n \geq 1). \qquad (5.10)$$

The inequality (5.10) yields

$$\sigma \leq \left\{ \frac{\nu(n) - 2p(p+n)(1-\alpha)^2}{\nu(n) - 2p^2\beta(1-\alpha)^2} \right\}$$

where $\nu(n)$ given by (4.2), and this inequality gives the required result.

Theorem 5.3. The class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ is convex.

Proof. Suppose that the functions $f_i(z)(i=1,2)$ defined by (1.5) are in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Then it is sufficient to show that the function

$$h(z)=\xi f_1(z)+(1-\xi)f_2(z) \qquad ,(0\le\xi\le1) \tag{5.11}$$

or, equivalently

$$h(z)=z^p-\sum_{n=1}^{\infty}\{\xi a_{p+n,1}+(1-\xi)a_{p+n,2}\}\,z^{p+n} \qquad ,(0\le\xi\le1) \tag{5.12}$$

is also in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$.

Now, from our hypothesis and Theorem 2.1, it follows readily that

$$\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^m[n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)\ \left(\xi a_{p+n,1}+(1-\xi)a_{p+n,2}\right)$$
$$\le p(1-\alpha)$$

which evidently proves Theorem 5.3.

6- Extreme Points

Theorem 6.1. Let

$$f_p(z)=z^p\,, \qquad (p\in\mathbb{N}) \tag{6.1}$$

and

$$f_{p+n}(z)=z^p-\frac{p(1-\alpha)}{\left(\frac{p+\delta n}{p}\right)^m[n+p(1-\beta\alpha)]\,\gamma_n(\lambda,\mu,\eta,p)}z^{p+n}\,, \qquad (p,n\in\mathbb{N}). \tag{6.2}$$

Then $f(z)\in\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ if and only if it can be expressed in the form:

$$f(z)=\sum_{n=0}^{\infty}\varepsilon_{p+n}\ f_{p+n}(z) \tag{6.3}$$

where

$$\varepsilon_{p+n}\ge0 \quad , \quad \sum_{n=0}^{\infty}\varepsilon_{p+n}=1 \tag{6.4}$$

Proof. Let

$$f(z)=\sum_{n=0}^{\infty}\varepsilon_{p+n}\ f_{p+n}(z)$$

$$= z^p - \sum_{n=1}^{\infty} \frac{p(1-\alpha)}{\left(\frac{p+\delta n}{p}\right)^m [n + p(1-\beta\alpha)]\, \gamma_n(\lambda,\mu,\eta,p)}\, \varepsilon_{p+n}\, z^{p+n}. \quad (6.5)$$

Then, in view of (6.4), it follows that

$$\sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n + p(1-\beta\alpha)]\, \gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \left\{ \frac{p(1-\alpha)}{\left(\frac{p+\delta n}{p}\right)^m [n + p(1-\beta\alpha)]\, \gamma_n(\lambda,\mu,\eta,p)}\, \varepsilon_{p+n} \right\}$$

$$= \sum_{n=1}^{\infty} \varepsilon_{p+n} = 1 - \varepsilon_p \le 1. \quad (6.6)$$

So, by Theorem 2.1, the function $f(z)$ belongs to the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$.

Conversely, let the function $f(z)$ defined by (1.4) belongs to the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$, then

$$a_{p+n} \le \frac{p(1-\alpha)}{\left(\frac{p+\delta n}{p}\right)^m [n + p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)} \quad ,(p,n \in \mathbb{N}). \quad (6.7)$$

Setting

$$\varepsilon_{p+n} = \frac{\left(\frac{p+\delta n}{p}\right)^m [n + p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)}\, a_{p+n} \quad ,(p,n \in \mathbb{N}), \quad (6.8)$$

and

$$\varepsilon_p = 1 - \sum_{n=1}^{\infty} \varepsilon_{p+n} \quad (6.9)$$

we can see that $f(z)$ can be expressed in the form (6.3). This completes the proof of Theorem 6.1.

Corollary 6.2. The extreme points of the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$ are the functions $f_p(z)$ and $f_{p+n}(z)$ given by (6.1) and (6.2) respectively.

7- Radii of Close-to-convexity, Starlikeness and Convexity

Theorem 7.1. Let the function $f(z)$ defined by (1.4) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Then $f(z)$ is p-valently close-to-convex of order σ $(0 \le \sigma < p)$ in the disk $|z| < r_3$ where

$$r_3 = \inf_{n\in\mathbb{N}}\left\{\frac{(p-\sigma)\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)(p+n)}\right\}^{1/n} \quad (7.1)$$

and $\gamma_n(\lambda,\mu,\eta,p)$ is given by (1.10). The result is sharp with the externaI function $f(z)$ given by (2.4).

Proof . It suffices to show that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right| \le p-\sigma \qquad ,(|z|<r_3) \quad (7.2)$$

Indeed we have

$$\left|\frac{f'(z)}{z^{p-1}}-p\right| \le \sum_{n=1}^{\infty}(p+n)a_{p+n}|z|^n \quad (7.3)$$

Hence (7.2) is true if

$$\sum_{n=1}^{\infty}(p+n)a_{p+n}|z|^n \le p-\sigma,$$

or

$$\sum_{n=1}^{\infty}\frac{(p+n)}{(p-\sigma)}a_{p+n}|z|^n \le 1. \quad (7.4)$$

By Theorem 2.1, (7.4) is true if

$$\frac{(p+n)}{(p-\sigma)}|z|^n \le \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \quad , \quad (n\in\mathbb{N}). \quad (7.5)$$

Solving (7.5) for $|z|$, we get the desired result (7.1).

Theorem 7.2. Let the function $f(z)$ defined by (1.4) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Then $f(z)$ is p-valently starlike of order σ $(0\le\sigma<p)$ in the disk $|z|<r_4$, where

$$r_4 = \inf_{n\in\mathbb{N}}\left\{\frac{(p-\sigma)\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)(p+n-\sigma)}\right\}^{1/n} \quad (7.6)$$

and $\gamma_n(\lambda,\mu,\eta,p)$ given by (1.10). The result is sharp with the externaI function $f(z)$ given by (2.4).

Proof. It suffices to show that

$$\left|\frac{z\,f'(z)}{f(z)}-p\right|\le p-\sigma \qquad ,(|z|<r_4) \tag{7.7}$$

Indeed we have

$$\left|\frac{zf'(z)}{f(z)}-p\right| = \left|\frac{-\sum_{n=1}^{\infty} n a_{p+n} z^n}{1-\sum_{n=1}^{\infty} a_{p+n} z^n}\right| \le \frac{\sum_{n=1}^{\infty} n a_{p+n} |z|^n}{1-\sum_{n=1}^{\infty} a_{p+n} |z|^n} \tag{7.8}$$

Hence (7.7) is true if

$$\sum_{n=1}^{\infty} n a_{p+n}\,|z|^n \le (p-\sigma)-\sum_{n=1}^{\infty}(p-\sigma)a_{p+n}\,|z|^n, \tag{7.9}$$

that is, if

$$\sum_{n=1}^{\infty}\frac{(p+n-\sigma)}{(p-\sigma)}a_{p+n}\,|z|^n \ \le 1 \tag{7.10}$$

By Theorem 2.1, (7.10) is true if

$$\frac{(p+n-\sigma)}{(p-\sigma)}\ |z|^n \le \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \quad , \qquad (n\in\mathbb{N}). \tag{7.11}$$

Solving (7.11) for $|z|$, we get the desired result (7.6).

Theorem 7.3. Let the function $f(z)$ defined by (1.4) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Then $f(z)$ is p-valently convex of order σ $(0\le\sigma<p)$ in the disk $|z|<r_5$, where

$$r_5=\inf_{n\in\mathbb{N}}\left\{\frac{(p-\sigma)\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{(p+n)(1-\alpha)(p+n-\sigma)}\right\}^{1/n} \tag{7.12}$$

and $\gamma_n(\lambda,\mu,\eta,p)$ given by (1.10). The result is sharp with the extermal function $f(z)$ given by (2.4).

Proof. It suffices to show that

$$\left|1+\frac{z\,f''(z)}{f'(z)}-p\right|\le p-\sigma \quad ,(|z|<r_5) \tag{7.13}$$

Indeed we have

$$\left|1+\frac{z\,f''(z)}{f'(z)}-p\right| = \left|\frac{-\sum_{n=1}^{\infty} n(p+n)a_{p+n}\,z^n}{p-\sum_{n=1}^{\infty}(p+n)\,a_{p+n}\,z^n}\right| \le \frac{\sum_{n=1}^{\infty} n(p+n)a_{p+n}\,|z|^n}{p-\sum_{n=1}^{\infty}(p+n)a_{p+n}\,|z|^n} \tag{7.14}$$

Hence (7.13) is true if

$$\sum_{n=1}^{\infty} n(p+n)\, a_{p+n}\, |z|^n \leq p(p-\sigma) - \sum_{n=1}^{\infty} (p-\sigma)(p+n) a_{p+n}\, |z|^n \tag{7.15}$$

or

$$\sum_{n=1}^{\infty} \frac{(p+n)(n+p-\sigma)}{p(p-\sigma)} a_{p+n}\, |z|^n \leq 1 \tag{7.16}$$

By Theorem 2.1, (7.16) is true if

$$\frac{(p+n)(n+p-\sigma)}{(p-\sigma)}\, |z|^n \leq \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{(1-\alpha)} \quad ,(n \in \mathbb{N}). \tag{7.17}$$

Solving (7.17) for $|z|$, we get the desired result (7.12).

8- Class-Preserving Integral Operators

We prove that the integral operator $J_{c,p}$ defined by (1.14) is indeed a class- preserving operator for the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$.

Theorem 8.1. Let the function $f(z)$ defined by (1.4) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Also let $c > -p$. Then the function $F(z)$ defined by (1.14) is also in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$.

Proof . from (1.14) and (1.4), it easily seen that

$$F(z) = z^p - \sum_{n=1}^{\infty} A_{p+n} z^{p+n}, \tag{8.1}$$

where

$$A_{p+n} = \left(\frac{c+p}{c+p+n}\right) a_{p+n} \,, \qquad (n \in \mathbb{N}; c > -p). \tag{8.2}$$

Since $c > -p$, we have

$$0 \leq A_{p+n} < a_{p+n} \,, \qquad (n \in \mathbb{N}), \tag{8.3}$$

which, in view of Theorem 2.1, immediately yields Theorem 8.1.

Remark 2. Letting $c = 1 - p$ in Theorem 8.1, we obtain the following result.

Corollary 8.2. Let the function $f(z)$ defined by (1.4) be in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Then

$$G(z) = z^{p-1} \int_0^z \frac{f(t)}{t^p}\, dt \tag{8.4}$$

is also in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$.

Theorem 8.3. Let $c > -p$. Also let $F(z)$ be in class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$. Then the function $f(z)$ given by (1.14) is p-valent in the disk $|z| < r_6$ where

$$r_6 = \inf_{n\in\mathbb{N}} \left\{ \frac{\left(\frac{p+\delta n}{p}\right)^m (c+p)[n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{(p+n)(c+p+n)(1-\alpha)} \right\}^{1/n} \tag{8.5}$$

Proof. Assuming that

$$F(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n}, \qquad (b_{p+n} \geq 0, p \in \mathbb{N}), \tag{8.6}$$

from (1.14), we get

$$f(z) = \frac{z^{1-c}}{c+p}\frac{d}{dz}\left(z^c F(z)\right) = z^p - \sum_{n=1}^{\infty}\left(\frac{c+p+n}{c+p}\right) b_{p+n} z^{p+n}, \quad (c > -p). \tag{8.7}$$

In order to prove the result, it suffices to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p, \qquad (|z| < r_6). \tag{8.8}$$

Indeed we have

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| = \left|-\sum_{n=1}^{\infty}(p+n)\left(\frac{c+p+n}{c+p}\right) b_{p+n} z^n\right|$$

$$\leq \sum_{n=1}^{\infty}(p+n)\left(\frac{c+p+n}{c+p}\right) b_{p+n}|z|^n,$$

which yields the desired inequality in (8.8), provided that

$$\sum_{n=1}^{\infty}\frac{(p+n)(c+p+n)}{p(c+p)} b_{p+n}|z|^n \leq 1. \tag{8.9}$$

But, since the function $F(z)$ defined by (8.6) is in the class $\mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$, Theorem 2.1 gives us

$$\sum_{n=1}^{\infty}\frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} b_{p+n} \leq 1. \tag{8.10}$$

Thus the inequality (8.9), and hence also the inequality (8.8), will hold true if

$$\frac{(p+n)(c+p+n)}{p(c+p)}|z|^n \leq \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} \qquad (n \in \mathbb{N}),$$

that is, if

$$|z| \leq \left\{ \frac{\left(\frac{p+\delta n}{p}\right)^m (c+p)[n+p(1-\beta\alpha)]\gamma_n(\lambda,\mu,\eta,p)}{(p+n)(c+p+n)(1-\alpha)} \right\}^{1/n} \quad (n \in \mathbb{N}),$$

which leads us precisely to the main assertion of Theorem 8.3.

9- Integral Means Inequalities

Applying Lemma 1.3, we prove the following theorem.

Theorem 9.1. Let $\tau > 0;\ \lambda \geq 0;\ \mu < p+1;\ \eta \geq \lambda\left(1-\frac{p+2}{\mu}\right);\ \delta \geq 0;\ m \in \mathbb{N}_0;\ 0 < \beta \leq 1;\ 0 \leq \alpha < 1;\ \lambda,\mu,\eta \in \mathbb{R}$ and $p \in \mathbb{N}$. If $f(z) \in \mathcal{M}_p^{m,\lambda,\mu,\eta,\delta}(\beta,\alpha)$, then for $z = re^{i\theta}$ and $0 < r < 1$, we have

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\tau d\theta \leq \int_0^{2\pi} \left|f_{p+1}(re^{i\theta})\right|^\tau d\theta \qquad (9.1)$$

where

$$f_{p+1}(z) = z^p - \frac{p(1-\alpha)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)]\,\gamma_1(\lambda,\mu,\eta,p)} z^{p+1} \qquad (9.2)$$

and $\gamma_n(\lambda,\mu,\eta,p)$ given by (1.10).

Proof. Let $f(z)$ of the form (1.4) and $f_{p+1}(z)$ of the form (9.2), then we must show that

$$\int_0^{2\pi} \left|1-\sum_{n=1}^{\infty} a_{p+n} z^n\right|^\tau d\theta \leq \int_0^{2\pi} \left|1 - \frac{p(1-\alpha)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)]\,\gamma_1(\lambda,\mu,\eta,p)} \mathrm{z}\right|^\tau d\theta.$$

By Lemma 1.3, it suffices to show that

$$1-\sum_{n=1}^{\infty} a_{p+n} z^n \prec 1 - \frac{p(1-\alpha)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)]\,\gamma_1(\lambda,\mu,\eta,p)} \mathrm{z}$$

Setting

$$1-\sum_{n=1}^{\infty} a_{p+n} z^n = 1 - \frac{p(1-\alpha)}{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)]\,\gamma_1(\lambda,\mu,\eta,p)} w(z). \qquad (9.3)$$

From (9.3) and (2.1), we obtain

$$|w(z)| = \left| \sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta}{p}\right)^m [1+p(1-\beta\alpha)]\, \gamma_1(\lambda,\mu,\eta,p)}{p(1-\alpha)} a_{p+n} z^n \right|$$

$$\leq |z| \sum_{n=1}^{\infty} \frac{\left(\frac{p+\delta n}{p}\right)^m [n+p(1-\beta\alpha)]\, \gamma_n(\lambda,\mu,\eta,p)}{p(1-\alpha)} a_{p+n} \leq |z| < 1.$$

This completes the proof of the Theorem 9.1.

Remark 3. Letting $p = 1, \lambda = \mu = \delta = 0$ and $\beta = 1$, in Theorem 9.1, we get the integral means inequality for the class $T^*(\alpha)$.

Corollary 9.2. Let $\tau > 0$. If $f(z) \in T^*(\alpha)$, then for $z = re^{i\theta}$ and $0 < r < 1$, we have

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\tau} d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^{\tau} d\theta \qquad (9.4)$$

where

$$f_2(z) = z - \frac{1-\alpha}{2-\alpha} z^2 \quad . \qquad (9.5)$$

Remark 4. If we take $\alpha = 0$ in $T^*(\alpha)$ of Corollary 9.2, we obtain the result proved by Silverman [9].

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