Classical Comparison of Numerical Methods for Solving Differential Equations of Fractional Order

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Abstract
In this paper, a numerical method is presented for finding the solution of differential equations. The main objective is to find the approximate solution of fractional differential equation of order α . This work is a comparison of some available numerical methods for solving linear "nonlinear" DEqs. of fractional order. However, all the previous works avoid the term of fractional derivative and handle them as a restricted variation. The present study shows that when these methods are applied to linear "nonlinear" DEqs. of fractional order, they have different convergence and approximation error.

Keywords: Fractional Calculus, Caputo fractional differential equations, Picard iteration, Gauss-Seidel method, Variationaliteration method.

1. Introduction

In recent years, the theory and applications of fractional equations were presented research topics in applied sciences; such as applied mathematics, physics, mathematical biology and engineering. The rule of fractional derivative is not unique date.Over the Past decade the development of numerical methods used for finding solutions of ordinary fractional differential equations containing derivatives of integer and non-integer order. There have been several algorithms published for producing approximate solutions for fractional differential equations.

The developments of theory and applications for approximate solutions of fractional differential equations have been completed. We refer to the articles, which work by authors as Diethelm, Ford [see; 8, 9, 10, 11,12,13].

The approximations and numerical techniques for differential equations of fractional order have been main objective in researches. Hence, there are some papers discussing numerical methods for solving fractional differential equations. Also, most the fractional equations do not have exact analytic solution. Consequently, we must used approximate and numerical techniques.

Recently, the analytical approximate solution for linear fractional differential equations with initial conditions has been used in [18]. The applications of methods for fractional equations was extended by authors in [14,18]

In this paper, we study the numerical approximate solution for linear differential equations of fractional order:

$$P(t)D^{\alpha}u(t)+a_m u^{(m)}(t)+a_{m-1}u^{(m-1)}(t)+...+a_2u''(t)+a_1u'(t)+a_0u(t)=f(t) \qquad (1.1)$$

Subject to initial conditions:

$$u^{(i)}(0)=\beta_i \quad ,i=1,2,...,m-1 \tag{1.2}$$

where $a_i,\beta_i \in \mathbb{R}$ are constants $t\in[0,T]$; $m-1<\alpha<m$; $f: J\times R\times R\to R$ is a continuous function,and $P(t)$ is known function. Here the notation D^{α} is used for Capote fractional derivative.

Diethelm, to solve the linear and non-linear differential equations recently used methods are Predictor-Corrector method [11], Adomain decomposition method [19, 20, 22], Homotopy Perturbation Method [12] Variational Iteration Method [15], in [21] the author using differential transform method to solving systems of fractional differential equations.

The approximate solutions have been obtained via several classes of fractional differential equations, where in [17] introduced discussing for approximate an ordinary fractional differential equation by the integer order differential equation with a small parameter and permits to find their approximate symmetries.

In this article, we study fractional differential equations associated to the a derivative. Such kind of equations appears in many problems. In particular, we have find a fractional differential equation related to the classical Gauss-Seidel method [3], and then comparison with the variation iteration method [23], which is confirmed through some examples.

The purpose of this study is to introduce approximate solutions for fractional differential of order α , $\alpha>0$ equations by using modified Picard iteration with Gauss-Seidel technique, which proposed by he [3] was successfully applied to solving linear (nonlinear) system of ordinary differential equations with initial conditions.

2. Definitions and properties in fractional calculus

In this section, we consider the main definitions of fractional derivatives of order α , $\alpha>0$, The Caputo and the Riemann-Liouville fractional derivatives [4] are both used here Whereasin mathematical treatises on fractional differential equations the Riemann-Liouville approach to the notion of the fractional derivative of order α . we begin by introducing the basic definitions:

Definition 2.1.A real function $f(t), t>0$, is said to be in the space C_{μ}, $\mu\in\mathbb{R}$, if there exists a real number $p>\mu$ such that $f(t)=t^{p}f_1(t)$, where $f_1(t)\in C[0,+\infty)$, it 's clearly, $C_{\mu}\subset C_{\beta}$ if $\beta\le\mu$.

Definition 2.2.A function $f(t), t>0$, is said to be in the space C_{μ}^{m} if $f^{(m)}\in C_{\mu}$ for $m\in\mathbb{N}\cup\{0\}$.

Definition 2.3. The Riemann-Liouville fractional integral operator of order $\alpha>0$ of a function $f\in C_{\mu}, \mu>1$, is defined as:

$$ {}_0I_t^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds, \quad \alpha>0, \quad t>0 \tag{2.1}$$

$${}_0I_t^{\alpha}f(t)=f(t)$$

where $\Gamma(.)$ is the Gamma function.

Definition 2.4. The fractional derivative of $f\ (t)$ in the Caputo derivative is defined as follows:

$$ {}_0^cD_t^\alpha f\ (t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t(t-s)^{m-\alpha-1}\frac{d^m}{ds^m}f\ (s)ds \qquad (2.2)$$

where $m-1<\alpha\leq m,\ m\in\mathbb{N},\ \ f\ \in C_1^m$, we rewrite last formula as the form:

$$ {}_0^cD_t^\alpha f\ (t)=\begin{cases} I^{m-\alpha}f^{(m)}(t) & ,\ m-1<\alpha<m,\ \ m\in\mathbb{N} \\ \dfrac{d^m}{dt^m}f\ (t) & ,\ \alpha=m \end{cases} \qquad (2.3)$$

Hence, we have the following properties for $\mathrm{f}\in\mathrm{C}_\mu$ and $\mu\geq-1$ have been proved; refer to the works [1, 2, 4, 6, 7, 16]:

1- ${}_0I_t^\alpha t^k=\dfrac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}t^{\alpha+k},\qquad \alpha>0,\ k\in\mathbb{N}\cup\{0\},\quad t>0$

2- ${}_0^cD_t^\alpha I_t^0 f\ (t)=f\ (t)$

3- ${}_0^cD_t^\alpha I_t^\alpha f\ (t)=f\ (t)-\sum_{k=0}^{n-1}f\ (0^+)\dfrac{t^k}{k!}\ ,\qquad t>0$

4- ${}_0^cD_t^\beta f\ (t)={}_0I_t^{\alpha-\beta}\ {}_0^cD_t^\alpha f\ (t)\ ,\qquad \alpha,\beta>0$

The existence, uniqueness, and structural stability of solutions of nonlinear differential equations of fractional order. It had been discussed in [10].

Theorem 2.1 (existence). Assume that $\mathfrak{D}:=[0,t^*]\times[u_0{}^{(0)}-\varepsilon,u_0{}^{(0)}+\varepsilon]$with some $t^*>0$ and some $e>0$, and let the function $f:\mathfrak{D}\to\mathbb{R}$, be continuous. Furthermore, define $x:=\min\left\{t^*,\ (\varepsilon\Gamma(\alpha+1)/\ ||f||_\infty)^{1/\alpha}\right\}$. Then, there exists a function $u:[0,x]\to\mathbb{R}$solving the initial value problem

$D^a(u-T_{m-1}[u])t=f\ (t,u(t)),\ u^{(j)}(0)=u_0^{(j)},j=0,1,2,...,m-1,$

where $T_{m-1}[u]$ is Taylor Polynomial of order $m-1$ for u.

Theorem 2.2(uniqueness). Assume that $\mathfrak{D}:=[0,t^*]'\ [u_0^{(0)}-e,u_0^{(0)}+e]$ with some $t^*>0$ and some$\varepsilon>0$. Furthermore, let the function$f:\mathfrak{D}\to\mathbb{R}$ be bounded on $\mathfrak{D}$ and fulfill a Lipschitz condition with respect to the second variable; i.e.,

$$|f\ (t,u)-f\ (t,u^*)|\text{£}\ L\,|u-u^*|\ \ with\ \ some\ \ cons.\ \ L>0$$

For the proofs of these two theorems, which was proved by applying the integral operator of order α, given by (see; [10])

$$ {}_0I_t^\alpha f\ (t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\ (s)ds \qquad (2.4)$$

3. Material and methods

In this section we will extend iteration method of fractional calculus, we review the classical method of successive approximation "Picard iteration", firstly, and so we use modified Picard iteration with Gauss-Seidel technique are given in [3].

Therefore, rewrite fractional differential equation (1) as the system of differential equations of first order, then equation (1) is transformed into the following ; let

$$u_1(t)=u(t)$$

$$\begin{aligned} &u_1' = u_2 \\ &u_2' = u_3 \\ &\vdots \\ &u_1' = f(t) - P(t)D^{\alpha}u(t) - a_m u_m(t) - a_{m-1}u_{m-1}(t) - \ldots - a_2u_3(t) - a_1u_2(t) - a_0u_1(t) \end{aligned} \tag{3.1}$$

subject to initial conditions:

$$u_i(0) = \beta_i, \quad i = 1,2,\ldots,m \tag{3.2}$$

Accordingly, the Picard iteration method for system of differential equations (3.1) is obtained by the replacement of every equation in (3.1) by using Gauss-Seidel technique, the result takes the form:

$$\begin{aligned} u_{1,n} &= u_{1,0} + \int_0^t u_{2,n-1}(s)ds \\ u_{2,n} &= u_{2,0} + \int_0^t u_{3,n-1}(s)ds \qquad n = 1,2,\ldots \\ &\vdots \\ u_{m,n} &= u_{m,0} + \int_0^t (f(s) - P(s)D^{\alpha}u_{1,n}(s) - a_m u_{m,n-1}(s) - a_{m-1}u_{m-1,n}(s) - \ldots - a_1u_{2,n}(s) - a_0u_{1,n}(s))ds \end{aligned} \tag{3.3}$$

subject to initial conditions :

$$u_i(0) = \beta_i, \quad i = 1,2,\ldots,m$$

Consequently, the variational iteration method of fractional differential equation (1.1) with initial conditions(2) can be constructed as the form(see,[23]):

$$\begin{aligned} u_n = u_{n-1}(t) + \int_0^t \lambda(P(s)D_0^{\alpha}u_{n-1}(s) + a_m D_t^m u_{n-1}(s) + a_{m-1}D_t^{m-1}u_{n-1}(s) + \ldots \\ \ldots + a_1 D_t u_{n-1}(s) + a_0 u_{n-1}(s) - f(s))ds \end{aligned} \tag{3.4}$$

$$t \ge 0, \quad n = 1,2,\ldots$$

where $D_t^m = \frac{d^m}{dt^m}$, and λ is a general Lagrange multiplier. If we repeat the above procedure, we have numerical solutions of fractional differential equations for (1.1).

4. Illustrative Examples

In the following examples, we consider numerical solutions of fractional differential equations of order α , to demonstrate the effectiveness of the method.

Example 1. We consider the following fractional differential equation

$$u'' - D^{\frac{1}{2}}u - u' = 2 - 2t - \frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}} \tag{4.1}$$

with initial conditions $u(0)=u'(0)=0$ $, u''(0)=2,$ $0 \le t \le 1;$
The corresponding system takes the form:

$$u_1' = u_2 \qquad u_1(0)=0$$
$$u_2' = 2-2t-\frac{8}{3\sqrt{\pi}}t + u_2 + D^{\frac{1}{2}}u_1 \qquad ,u_2(0)=0 \tag{4.2}$$

Accordingly, the classical Picard iteration method takes the form

$$u_{1,n} = 1+\int_0^t u_{2,n-1}ds, \qquad n=1,2,...$$
$$u_{2,n} = -1+\int_0^t (2-2s-\frac{8}{3\sqrt{\pi}}s^{\frac{3}{2}} + u_{2,n-1} + D^{\frac{1}{2}}u_{1,n-1})ds \tag{4.3}$$

Hence, the corresponding modified Picard iteration with Gauss-Seidel technique the integral will be came:

$$u_{1,n} = 1+\int_0^t u_{2,n-1}ds, \qquad n=1,2,...$$
$$u_{2,n} = -1+\int_0^t (2(1-s)-\frac{8}{3\sqrt{\pi}}s^{\frac{3}{2}} + u_{2,n-1} + D^{\frac{1}{2}}u_{1,n})ds \tag{4.4}$$

Table 1: shows the approximate solutions for Eq. (4.1) obtained for different methods. The results showed that the modified Picard iteration with Gauss-Seidel method is remarkably effective and performing is very easy. Additionally, it has more accuracy than Picard method and variation iteration method.

t_i	*Exact*	*PI*	*PI with GSM*	$\lvert u - u_{P:GS} \rvert$	*VIM*	$\lvert u - u_{VIM} \rvert$
0.0	0.00	0.00	0.00	0.00	0.00	0.00
0.1	0.01	0.01	0.01	$4.01971' \ 10^{-12}$	0.01	$3.46178' \ 10^{-13}$
0.2	0.04	0.0400021	0.04	$9.41858' \ 10^{-10}$	0.03999999996	$3.75077' \ 10^{-11}$
0.3	0.09	0.0900205	0.09	$2.35103' \ 10^{-8}$	0.08999999924	$7.56081' \ 10^{-10}$
0.4	0.16	0.160106	0.16	$2.33746' \ 10^{-7}$	0.15999999207	$7.93355' \ 10^{-9}$
0.5	0.25	0.250382	0.250001	$1.39986' \ 10^{-6}$	0.24999994397	$5.60354' \ 10^{-8}$
0.6	0.36	0.361092	0.360006	$6.07411' \ 10^{-6}$	0.35999970314	$2.96859' \ 10^{-7}$
0.7	0.49	0.492655	0.490021	0.0000210781	0.48999873765	$1.26235' \ 10^{-6}$
0.8	0.64	0.64574	0.640062	0.0000620574	0.63999547869	$4.52131' \ 10^{-6}$
0.9	0.81	0.821332	0.810161	0.00016105	0.80998587172	0.0000141283
1.0	1.00	1.02082	1.00038	0.000378167	0.999960462	0.0000395377

Table 1: shows the approximate solutions for Eq. (4.1)
eighth iterations which was obtained for different methods

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Now, we compare the seventh and eighth iterations Picard iteration, modified Picard with Gauss-Seidel method and variation iteration method with the exact solution on the graphs. These comparisons can be seen in figures 1, 2.

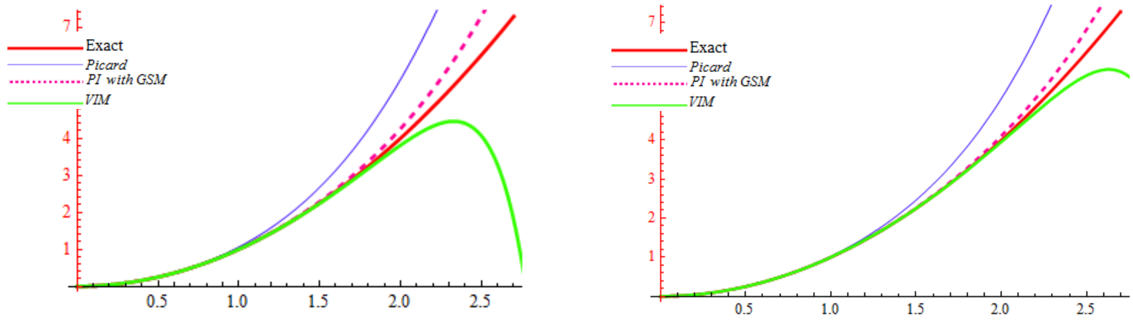

Figure 1: Comparison of approximate results for different methods

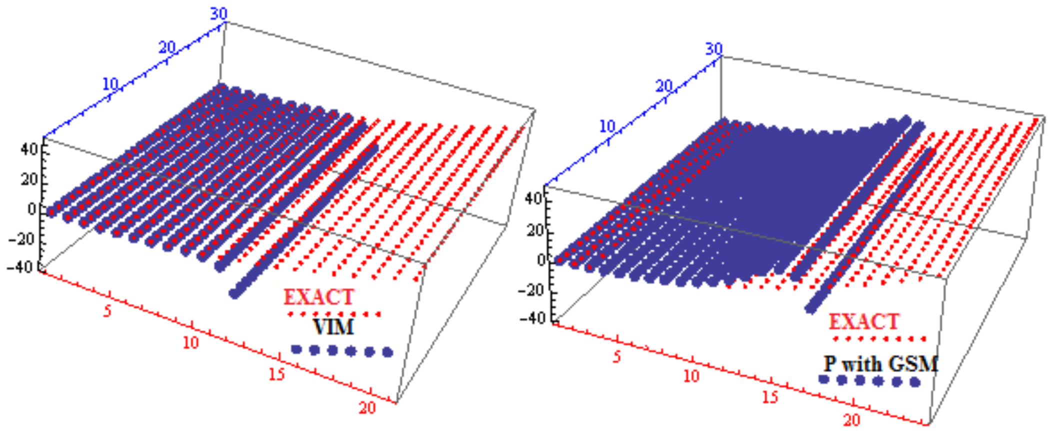

Figure 2: Comparison of eighth iteration approximate results of modified GSM and VIM with the exact solutions for eq. (4.1).

Example 2. We consider the following fractional differential equationof order $\alpha = \frac{1}{2}$

$$u'' + D^{\frac{1}{2}}u + u' = t + \frac{4}{3\sqrt{\pi}}t^{\frac{3}{2}} - \frac{2}{\sqrt{\pi}}\sqrt{t} \tag{4.5}$$

With initial conditions $u(0) = 0,\ u'(0) = -1\ , u''(0) = 1,\ 0 \le t \le 1;$

Applying the modified Picard iteration with Gauss-Seidel method, we get the following corresponding system:

$$\begin{aligned} u_{1,n} &= u_{1,0} + \int_0^t u_{2,n-1}\,ds \qquad u_1(0) = 0 \\ u_2 &= u_{1,0} + \int_0^t \left(s + \frac{4}{3\sqrt{\pi}}s^{\frac{3}{2}} - \frac{2}{\sqrt{\pi}}\sqrt{s} - u_{2,n-1} - D^{\frac{1}{2}}u_{1,n}\right) \qquad u_2(0) = -1 \end{aligned} \tag{4.6}$$

Table 2: shows the approximate solutions for Eq. (4.5) obtained for different methods. The results showed that the modified Picard iteration with Gauss-Seidel method is remarkably effective and performing is very easy. additionally, it has more accuracy than Picard method and variation iteration method.

t_i	*Exact*	*PI*	*PI with GSM*	$\lvert u - u_{P:GS} \rvert$	*VIM*	$\lvert u - u_{VIM} \rvert$
0.0	0.0	0.0	0.00	0.00	0.0	0.0
0.1	-0.095	-0.095	-0.095	3.94032×10^{-13}	-0.0942578	0.000742164
0.2	-0.18	-0.18	-0.18	1.23321×10^{-10}	-0.176293	0.00370736
0.3	-0.255	-0.255002	-0.255	3.6863×10^{-9}	-0.246077	0.00892336
0.4	-0.32	-0.320009	-0.32	4.18749×10^{-8}	-0.304239	0.0157607
0.5	-0.375	-0.375026	-0.375	2.79196×10^{-7}	-0.351864	0.0231364
0.6	-0.42	-0.420059	-0.420001	1.32694×10^{-6}	-0.390581	0.0294194
0.7	-0.455	-0.455116	-0.455005	4.98825×10^{-6}	-0.422926	0.0320741
0.8	-0.48	-0.4802	-0.480016	0.0000157847	-0.453061	0.0269391
0.9	-0.495	-0.495315	-0.495044	0.000043773	-0.488049	0.00695145
1.0	-0.5	-0.500463	-0.500109	0.000109352	-0.539968	0.0399684

Table 2: shows the approximate solutions for Eq. (4.5)
seventh iteration which was obtained for different methods.

The compare of the sixth and seventh iterations for Picard iteration, modified Picard with Gauss-Seidel method and variation iteration method with the exact solution appear on the graphs. These comparisons can be seen in figures 3,4. The results are in good agreement with the results of the exact solutions.

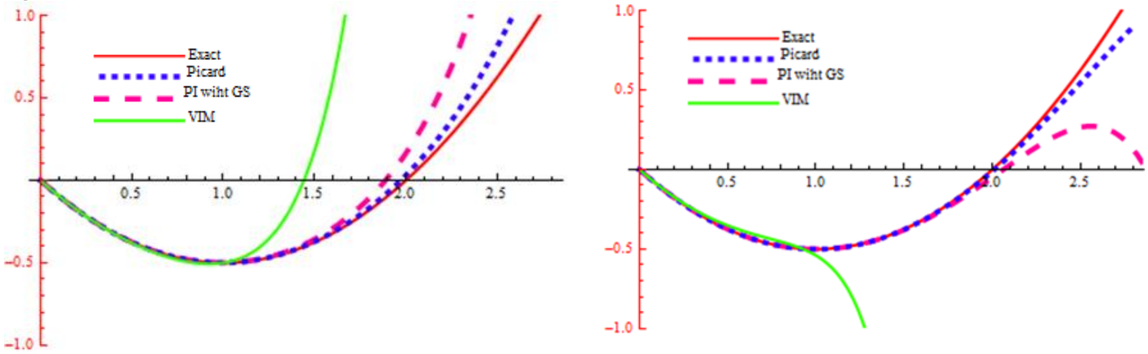

Figure 3 Comparison of approximate results for different methods

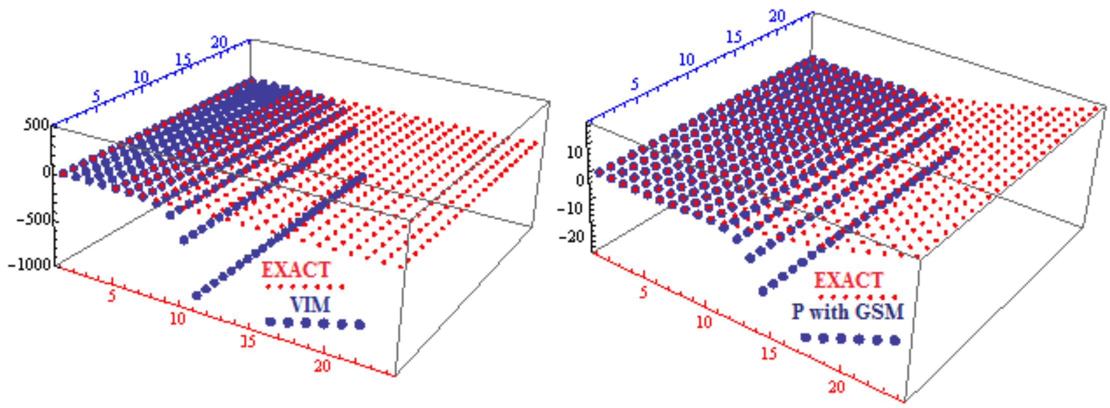

Figure 4 Comparison of seventh iteration approximate results of modified GSM and VIM with the exact solutions for eq. (4.5).

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Example 3. We consider the following fractional differential equationof order $\alpha = 3/2$

$$u'' - tD^{\frac{3}{2}}u + u' = 3t^2 - \frac{8}{\sqrt{\pi}}t^{\frac{5}{2}} + \frac{t^{\frac{3}{2}}}{6} + 6t - \frac{\sqrt{\pi}}{12}t - \frac{\sqrt{\pi}}{12}t \qquad (4.7)$$

with initial conditions $u(0) = 0,\ u'(0) = 0\ ,\ u''(0) = \frac{-\sqrt{\pi}}{12}, 0 \le t \le 1;$

Applying the modified Picard iteration with Gauss-Seidel method, we get the following corresponding system:

$$u_{1,n} = u_{1,0} + \int_0^t u_{2,n-1}\,ds \qquad u_1(0) = 0 \quad ,\ u_2(0) = 0 \qquad (4.8)$$

$$u_{2,n} = u_{2,0} + \int_0^t (3s^2 - \frac{8}{\sqrt{\pi}}s^{\frac{5}{2}} + \frac{s^{\frac{3}{2}}}{6} + 6s - \frac{\sqrt{\pi}}{12}s - \frac{\sqrt{\pi}}{12}s - u_{2,n-1} - sD^{\frac{3}{2}}u_{1,n})ds$$

Table 3: shows the approximate solutions for Eq. (4.7) obtained for different methods. The results showed that the modified Picard iteration with Gauss-Seidel method is remarkably effective and performing is very easy.

t_i	*Exact*	*PI*	*PI with GSM*	$\lvert u - u_{P:GS} \rvert$	*VIM*	$\lvert u - u_{VIM} \rvert$
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.000261478	0.000261477	0.000261478	1.82736×10^{-13}	0.000261478	3.01696×10^{-14}
0.2	0.00504591	0.00504588	0.00504591	6.10811×10^{-12}	0.00504591	3.9601×10^{-12}
0.3	0.0203533	0.0203526	0.0203533	3.3046×10^{-10}	0.0203533	8.11533×10^{-11}
0.4	0.0521836	0.052177	0.0521836	6.51023×10^{-9}	0.0521836	8.56785×10^{-10}
0.5	0.106537	0.1065	0.106537	5.29252×10^{-8}	0.106537	6.12635×10^{-9}
0.6	0.189413	0.189261	0.189413	2.69198×10^{-7}	0.189413	3.26002×10^{-8}
0.7	0.306812	0.30632	0.306813	9.78959×10^{-7}	0.306812	1.3515×10^{-7}
0.8	0.464735	0.463394	0.464737	2.61633×10^{-6}	0.464734	4.37791×10^{-7}
0.9	0.66918	0.665988	0.669184	4.56682×10^{-6}	0.669179	1.01397×10^{-6}
1.0	0.926148	0.919333	0.926148	2.89022×10^{-7}	0.926147	8.00177×10^{-7}

Table 3: shows the approximate solutions for Eq. (4.7)
ninth iteration which was obtained for different methods.

We compare the eighth iteration and ninth iteration .For Picard iteration, modified Picard with Gauss-Seidel method and variation iteration method with the exact solution on the graphs. These comparisons can be seen in figures5,6. The results are in good agreement with the results of the exact solutions.

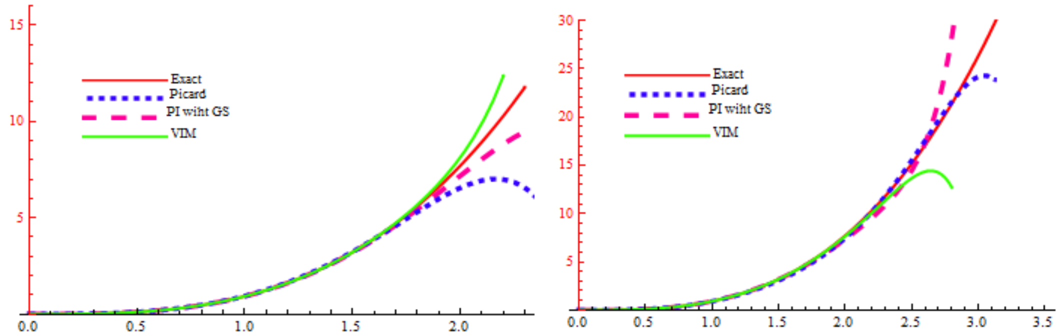

Figure 5 Comparison of approximate results for different methods

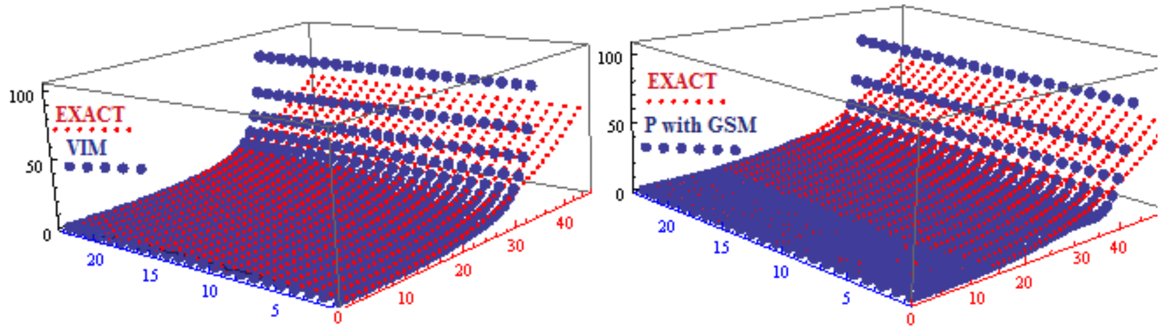

Figure 6 Comparison of ninth iteration approximate results of modified GSM and VIM with the exact solutions for eq. (4.7).

Example 4. We consider the following fractional differential equation of order $\alpha = 5/2$

$$u''' + t^2 D^{\frac{5}{2}} u - u'' = 2t - 2t + \frac{4}{\sqrt{\pi}} t^{\frac{5}{2}} \qquad (4.9)$$

with initial conditions $u(0) = 0, u'(0) = 0, u''(0) = 0, u'''(0) = 2$, $0 \le t \le 1$.

Applying the modified Picard iteration with Gauss-Seidel method, we get the following corresponding system:

$$\begin{aligned} u_{1,n} &= u_{1,0} + \int_0^t u_{2,n-1}\, ds \qquad u_1(0) = 0 \\ u_{2,n} &= u_{2,0} + \int_0^t u_{3,n-1}\, ds \qquad u_2(0) = 0 \\ u_{3,n} &= u_{3,0} + \int_0^t (2 - 2s + \frac{4}{\sqrt{\pi}} s^{\frac{5}{2}} + u_{3,n-1} - s^2 D^{\frac{5}{2}} u_{1,n}) ds \qquad , u_3(0) = 0 \end{aligned} \qquad (4.10)$$

Table 4: shows the approximate solutions for Eq. (4.9) obtained for different methods. The results showed that the modified Picard iteration with Gauss-Seidel method is remarkably effective and performing is very easy.

t_i	Exact	PI	PI with GSM	$\lvert u - u_{P:GS} \rvert$	VIM	$\lvert u - u_{VIM} \rvert$
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.000333333	0.000333455	0.000333333	1.18804´ 10^{-10}	0.000333333	1.75966´ 10^{-16}
0.2	0.00266667	0.0026707	0.00266668	1.69048´ 10^{-8}	0.00266667	9.24512´ 10^{-14}
0.3	0.009	0.00903215	0.00900028	2.78192´ 10^{-7}	0.009	2.7276´ 10^{-12}
0.4	0.0213333	0.0214765	0.0213352	1.90339´ 10^{-6}	0.0213333	1.99798´ 10^{-11}
0.5	0.0416667	0.0421287	0.0416746	7.96436´ 10^{-6}	0.0416667	1.01595´ 10^{-11}
0.6	0.072	0.073213	0.072024	0.0000240288	0.072	7.1958´ 10^{-10}
0.7	0.114333	0.117087	0.11439	0.0000566307	0.114333	6.15714´ 10^{-9}
0.8	0.170667	0.176271	0.170775	0.000108142	0.170667	3.06034´ 10^{-8}
0.9	0.243	0.253458	0.243168	0.000168016	0.243	9.82856´ 10^{-8}
1.0	0.333333	0.351508	0.333538	0.000204398	0.333333	9.28892´ 10^{-8}

Table 4 shows the approximate solutions for Eq. (4.9) eighth iteration which was obtained for different methods.

The comparison of the seventh and eighth iterations Picard iteration, modified Picard with Gauss-Seidel method and variation iteration method with the exact solution appear on the graphs. These comparisons can be seen in figures 7,8 The results are in good agreement with the results of the exact solutions.

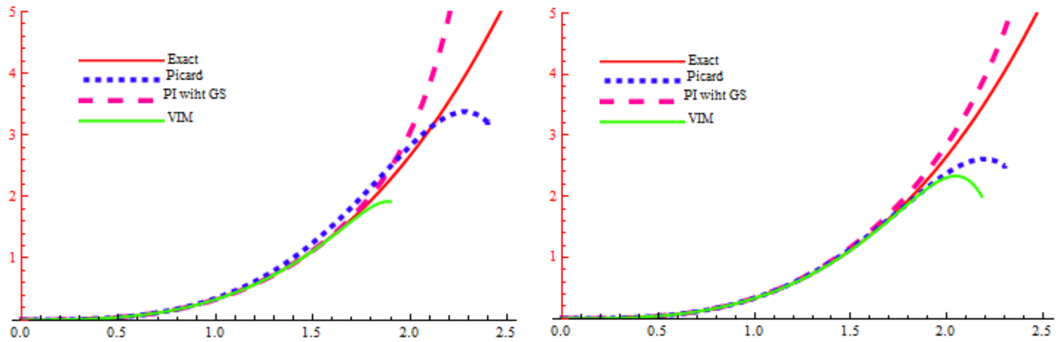

Figure 7 Comparison of approximate results for different methods

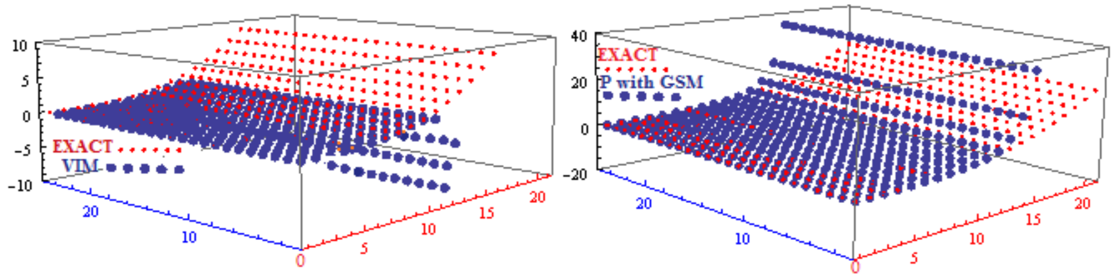

Figure 8 Comparison of eighth iteration approximate results of modified GSM and VIM with the exact solutions for eq. (4.9).

5. Conclusion

The fundamental goal of article is to construct the approximate solutions of fractional derivatives of order α. the aim has been achieved by using the classical Picard method, the modified Picard method and compared them with VIM to investigate the efficiency of improved Gauss-Seidel technique against classical Picard iteration and comparison to the VIM. Those methods are based on the numerical approximation of the fractional derivatives and integral in the continues time. Although several of the earlier papers rely on the smoothness of the solution to prove results on the rates of convergence. This is the classical approach from ordinary differential equations. However, for fractional equations even polynomial solutions may become non-smooth following fractional order differentiation. Therefore we explore briefly whether the form of the solution affects the performance of the method. Consequently, the modified method is a powerful and efficient technique for the solution linear fractional differential equations. It provides the analyst with an easily computable, readily verifiable and rapidly convergent sequence of analytic approximate functions for the solution, and also are relatively better as expected.

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