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Solution of Rational Difference Equations by Applying the Differential Transform Method

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الملخص

في السنوات الأخيرة، كُرِّس اهتمام كبير للطرق المطورة حديثاً لايجاد الحلول التحليلية لمعادلات الفرق الغير خطية. وإحدى هذه الطرق المعتمدة هي طريقة التحويل التفاضلي DTM والتي استُخدمت على نطاق واسع لحل العديد من المسائل الخطية واللاخطية. في هذه الورقة نبحث امكانية استخدام DTM لحل معادلات الفرق القياسية وتحديد مشاكل جديدة لهذا النوع.

Abstract

In recent years, much consideration has been devoted to the newly developed methods to construct analytic solutions of non-linear difference equations. One of these reliable methods is the Differential Transform Method DTM which has been used extensively to solve different types of linear and non-linear problems. In this paper, we explore the potential of using the DTM to solve rational difference equations and identify new open problems and conjectures of this kind.

Keywords: Rational Difference Equations; Differential Transform Method (DTM).

1. Introduction

The differential transform method obtains an analytical solution in the form of polynomial. It is different from the traditional high order Taylor's series method, which requires figurative opposition of the necessary derivative of the data functions. The Taylor series method is computationally expensive for large orders (Odibat et al.2010). The differential transform method is an alternative procedure for obtaining analytical Taylor series solution of the difference equations its main application concern with both linear and nonlinear initial value problems in electrical circuit analysis. With this technique, the given difference equation and related initial conditions are transformed into recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution. This method is useful for obtaining exact and approximate solutions of linear and nonlinear difference equations. In this paper, we explore the potential use of DTM to solve rational difference equations in different orders.

2. Definitions and Theorems

Definition 1: (Chen & Ho 1996). The differential transform of a function $y = y(x)$

is defined as follows:

$$Y(k)=\frac{1}{k!}\left[\frac{d^k}{dx^k}y(x)\right]_{x=x_0} \tag{1}$$

$y(x)$ is the original function and where it may be noted that upper case symbol $Y(k)$ is the transformed function of $y(x)$ by the DTM and the inverse transformation is defined by

$$y(x)=\sum_{k=0}^{\infty}(x-x_0)^k Y(k). \tag{2}$$

Using Eq.(1) and Eq.(2), one may write

$$y(x)=\sum_{k=0}^{\infty}\frac{(x-x_0)^k}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0} \tag{3}$$

Eq.(3) implies that the concept of differential transform is derived from the Taylor series expansion.

In actual applications the function is expressed by a truncated series and Eq.(2) can be written as

$$\varphi_N(x)=\sum_{k=0}^{N}Y(k)x^k.$$

A Table of the Fundamental Operations of One-Dimensional DTM

Original function	Transformed function
$y(x)=u(x)\pm v(x)$	$Y(k)=U(k)\pm V(k)$
$y(x)=\alpha u(x)$	$Y(k)=\alpha U(k)$
$y(x)=u(x)v(x)$	$Y(k)=\sum_{l=0}^{k}V(l)U(k-l)$
$y(x)=\frac{g(x)}{z(x)}$	$Y(k)=\frac{G(k)-\sum_{\ell=0}^{k-1}F(\ell)Z(k-\ell)}{Z(0)}$
$y(x)=\frac{du(x)}{dx}$	$Y(k)=(k+1)U(k+1)$
$y(x)=\frac{d^n u(x)}{dx^n}$	$Y(k)=\frac{(k+n)!}{k!}U(k+n)$
$y(x)=\int_{x_0}^{x}u(t)dt$	$Y(k)=(k+1)(k+2)\ldots(k+m)U(k+m)$

$y(x)=exp(\lambda x)$	$Y(k)=\dfrac{\lambda^k}{k!}$
$y(x)=sin(wx+\alpha)$	$Y(k)=(\dfrac{w^k}{k!})sin(\pi k/2+\alpha)$
$y(x)=cos(wx+\alpha)$	$Y(k)=(\dfrac{w^k}{k!})cos(\pi k/2+\alpha)$

Source: Abdel-Halim 2002; Odibat et al. 2010

We will use the subsequent theorems which have been proved before by (Arikoglu & Ozkol 2006).

Theorem 1 If $y(x)=x^n$ *then* $Y(k)=\delta(k-n)$, *where*

$$\delta(k-n)=\begin{cases}1, & k=n,\\ 0, & k\neq n,\end{cases}$$

Theorem 2 If $y(x)=g_1(x)g_2(x)\ldots g_{n-1}(x)g_n(x)$,*then*

$$Y(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}G_1(k_1)G_2(k_2-k_1)\ldots G_{n-1}(k_{n-1}-k_{n-2})\ G_n(k-k_{n-1}).$$

Theorem 3 If $y(x)=g(x+a)$,*then* $Y(k)=\sum_{h_1=k}^{N}\binom{h_1}{k}a^{h_1-k}G(h_1)$ *for* $N\to\infty$

Theorem 4 If $y(x)=g_1(x+a_1)g_2(x+a_2)$ *, then*

$$Y(k)=\sum_{h_1=0}^{k}\sum_{h_1=k_1}^{N}\sum_{h_2=k-k_1}^{N}\binom{h_1}{k_1}\binom{h_2}{k-k_1}a_1^{h_1-k_1}a_2^{h_2-k+h_1}G_1(h_1)G_2(h_2) \qquad for\ N\to\infty$$

Theorem 5 If $y(x)=g_1(x+a_1)g_2(x+a_2)\ldots g_{n-1}(x+a_{n-1})g_n(x+a_n)$,*then*

$$Y(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}\sum_{h_1=k_1}^{N}\sum_{h_2=k_2-k_1}^{N}\cdots\sum_{h_{n-1}=k_{n-1}-k_{n-2}}^{N}\ \sum_{h_n=k-k_{n-1}}^{N}\binom{h_1}{k_1}$$

$$\times\binom{h_2}{k_2-k_1}\cdots\binom{k_{n-1}}{k_{n-1}-k_{n-2}}\binom{h_1}{k-k_{n-1}}a_1^{h_1-k_1}a_2^{h_2-k_2+k_1}\ldots a_{n-1}^{h_{n-1}-k_{n-1}+k_{n-2}}$$

$$\times a_n^{h_n-k+k_{n-1}}G_1(h_1)G_2(h_2)\ldots G_{n-1}(h_{n-1})G_n(h_n) \qquad for\ \ N\to\infty$$

Theorem 6 If $y(x)=g_1(x+a_1)g_2(x+a_2)\dots g_{n-1}(x+a_{n-1})g_n(x+a_n)h_1(x)$

$\times h_2(x)\dots h_{m-1}(x)h_m(x)$, *then*

$$Y(k)=\sum_{k_{m+n-1}=0}^{k}\sum_{k_{m+n-2}=0}^{k_{m+n-1}}\dots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}\sum_{h_1=k_1}^{N}\sum_{h_2=k_2-k_1}^{N}\dots\sum_{h_{n-1}=k_{n-1}-k_{n-2}}^{N}\sum_{h_n=k_n-k_{n-1}}^{N}\binom{h_1}{k_1}\binom{h_2}{k_2-k_1}\dots$$

$$\times\binom{h_{n-1}}{k_{n-1}-k_{n-2}}\binom{h_n}{k_n-k_{n-1}}a_1^{h_1-k_1}a_2^{h_2-k_2+k_1}\dots a_{n-1}^{h_{n-1}-k_{n-1}+k_{n-2}}a_n^{h_n-k_n+k_{n-1}}G_1(h_1)G_2(h_2)\dots$$

$$\times G_{n-1}(h_{n-1})G_n(h_n)H_1(k_{n+1}-k_n)H_2(k_{n+2}-k_{n+1})\dots$$

$$\times H_{m-1}(k_{m+n-1}-k_{m+n-2})H_m(k-k_{m+n-1}) \qquad for \quad N\to\infty.$$

3. Technique of Solving Difference Equations by DTM

In this section we present the process of how difference equations can be solved by using our particular method. This technique of the solution is difficult to obtain without assist from computer programmers such as Maple or Matlab or other software. In this work we use Maple of version 12 for getting our results.

In 2006, Arikoglu & Ozkol have introduced for the first time the procedure of solving difference equations by using DTM as follows:

Generally, by considering the following difference equation

$$f\left(y,y(x),y(x+1),......,y(x+m)\right)=0, \tag{4}$$

Where f is a given function, m is some positive integer, and $x=0,1,2,....$ with initial conditions

$$y(b_i)=c_i, \qquad \text{for } i=0,1,2,3,\dots,n. \tag{5}$$

After applying the DTM on both sides in Eq.(4) by using the previous theorems , the following expression are obtained:

$$F\left[Y(0),Y(1),...,Y(N),k\right]=0, \tag{6}$$

where N is sufficiently large integer. Then, the initial conditions in Eq.(5) are transformed by using Eq.(2) as

$$\sum_{k=0}^{N}Y(k)(b_i-x_0)^k=c_i, \qquad for \quad i=1,2,3,\dots,n. \tag{7}$$

Then, in order to solve the difference Eq.(1) it is required the solution of coefficient $Y(0), Y(1), \ldots, Y(N)$.As a result, *N+1* equations for *N+1* must be obtained for *N*+1 unknowns.

As we notice there is n equations from Eq.(7) given us initial conditions and $N-n+1$ equations of the remaining equations that we should get from the transformed initial conditions in Eq.(7) for $k=0,1,2,\ldots,N-n$.

After applying the inverse transformation method in Eq.(3), the final solution series of Eq.(4) is as follows:

$$y(x)=Y(0)+Y(1)+Y(2)x^2+.....+Y(N)x^N+O(x^{N+1}) \qquad (8)$$

Definition 2 : (kulenovic & Ladas 2002). A first-order rational difference equation is nonlinear difference equation of the form

$$y(x+1)=\frac{\beta y(x)+\alpha}{By(x)+A}, \quad x=0,1,.... \qquad (9)$$

is called a *Riccati Difference Equation*, where the parameters α, β, A, B are real numbers and the initial condition $y(0)$ is real number such that the denominator is always positive.

To avoid generate cases, we will assume that

$$B \neq 0 \quad and \quad \beta A-\alpha B \neq 0 \qquad (10)$$

Indeed, when $B=0$, Eq.(9) is a linear equation and when $\beta A-\alpha B=0$,Eq.(9) reduces to

$$y(x)=\frac{\beta y(x)+\frac{\beta A}{B}}{By(x)+A}=\frac{\beta}{B} \quad for\ all \quad x \geq 0$$

Definition 3 : (Kuenovie & Ladas 2002). A second-order rational difference equation is a nonlinear difference equation of the form

$$y(x+1)=\frac{\alpha+\beta y(x)+\gamma y(x-1)}{A+By(x)+Cy(x-1)} \quad ,x=0,1,.... \qquad (11)$$

where the parameters $\alpha, \beta, \gamma, A, B, C$ are nonnegative real numbers and the initial conditions $y(-1)$ and $y(0)$ are arbitrary nonnegative real numbers such that

$$A+By(x)+Cy(x-1)>0 \quad \text{for all} \quad x \geq 0$$

Definition 4 : (Camouzis & Ladas 2008). Athird-order rational difference equation is a nonlinear difference equation of the form

$$y(x+1)=\frac{\alpha+\beta y(x)+\gamma y(x-1)+\delta y(x-2)}{A+By(x)+Cy(x-1)+Dy(x-2)} \quad ,x=0,1,.... \qquad (12)$$

where the parameters $\alpha, \beta, \gamma, \delta, A, B, C, D$, are nonnegative real numbers and the initial conditions $y(-2)$, $y(-1)$ and $y(0)$ are arbitrary nonnegative real numbers such that the denominator is always positive.

Definition 5: (Kulenovic & Ladas 2002).

(a) A solution $\{y(x)\}$ of Eq.(9),(11) and (12) is said to be *periodic* with period p if

$$y(x+p) = y(x) \qquad \text{for all} \quad x \geq -1 \tag{13}$$

(b) A solution $\{y(x)\}$ of Eqs.(8),(10) and (11) is said to be *periodic with prime period p*, or *a p-cycle* if it is periodic with period *p* and *p* is the least positive integer for which (13) holds.

4. Numerical Results

Example 1 (Camouzis & Ladas 2008). Show that every positive solution of the following first order- rational difference equation is periodic-2.

$$y(x+1) = \frac{1}{y(x)}, \quad x = 0,1,2,... \tag{14}$$

A function of the form

$$f(x) = \frac{g(x)}{z(x)}$$

Its transformed function by DTM is

$$F(k) = \frac{G(k) - \sum_{\ell=0}^{k-1} F(\ell) Z(k-\ell)}{Z(0)}, \tag{15}$$

and also, by using Theorems (1) and (3), the Differential Transform of Eq. (14) will be deduced as following:

Since

$$F(k) = \sum_{h=k}^{N} \binom{h}{k} Y(h),$$

$$G(k) = \delta(k), \; F(\ell) = \sum_{h=\ell}^{N} \binom{h}{\ell} Y(h),$$

$Z(k) = Y(k)$ and $Z(k-\ell) = Y(k-\ell)$

Therefore by Eq. (14) we have,

$$\sum_{h=k}^{N}\binom{h}{k}Y(h)=\frac{\delta(k)-\sum_{\ell=0}^{k-1}(\sum_{h=\ell}^{N}\binom{h}{\ell}Y(h))(Y(k-\ell))}{Y(0)} \tag{16}$$

After simplification we obtain

$$Y(0)\sum_{h=k}^{N}\binom{h}{k}Y(h)+\sum_{\ell=0}^{k-1}(\sum_{h=\ell}^{N}\binom{h}{\ell}Y(h))(Y(k-\ell))=\delta(k) \tag{17}$$

where *N* is sufficiently large integer.

As there is no any initial conditions *n=0* in Eq.(5) in this kind of examples we will consider *k=0, 1,.., N*. since *k=0,1,.....,N-n*

By taking *N*=3*,* the subsequent system of equations can be obtained from Eq. (17) for *k*=0, 1, 2, 3.

$$\begin{aligned}
&Y(0)(Y(0)+Y(1)+Y(2)+Y(3)) \;=\; 1\\
&Y(0)(Y(1)+2Y(2)+3Y(3))+(Y(0)+Y(1)+Y(2)+Y(3))Y(1) \;=\; 0\\
&Y(0)(Y(2)+3Y(3))+(Y(0)+Y(1)+Y(2)+Y(3))Y(2)+\\
&\qquad +(Y(0)+2Y(1)+3Y(2)+4Y(3))Y(1)=0\\
&Y(0)Y(3)+(Y(0)+Y(1)+Y(2)+Y(3))Y(3)+\\
&+(Y(0)+2Y(1)+3Y(2)+4Y(3))Y(2)+(Y(0)+2Y(1)+4Y(2)+7Y(3))Y(1) \;=\; 0
\end{aligned} \tag{18}$$

Solving the system of Eq. (18) we get the next:

$Y(0)=1,\; Y(1)=0,\; Y(2)=0,\; Y(3)=0,$

$Y(0)=\text{-}1,\; Y(1)=0,\; Y(2)=0,\; Y(3)=0,$ and

$Y(0)=\text{-}0.87300875,\; Y(1)=0.98117868,\; Y(2)=\text{-}1.4922638,\; Y(3)=0.23867218.$

Then using the inverse transformation rule at $x_0=0$ in Eq. (2), the subsequent three series solutions are obtained:

$$y_1(x)=1+O(x^4) \tag{19}$$

$$y_2(x)=-1+O(x^4) \tag{20}$$

$$y_3(x)=\text{-}0.87300875+0.98117868x\text{-}1.4922638x^2+0.23867218x^3+O(x^4) \tag{21}$$

It is apparent that solutions in Eqs. (19) and (20) satisfy the relation $y(x+2)=y(x)$ for all $x=0,1,2,...$

One can notice that the solution of Eq.(20) is negative for all *x*. Therefore, we do not consider this solution.

Since the solution in Eq.(19) is a positive solution of Eq.(13) as well as $y_1(x+2)=y_1(x)=1$ for all $x\geq -1$. Consequently, $y_1(x)=1+O(x^4)$ is a periodic solution with period 2.

In spite of the fact that, $y_2(x+2)=y_2(x)=-1$ is a periodic solution with period two for all $x\geq -1$, the solution is negative for all x . Accordingly, we pay no attention to this solution as well as the solution in Eq.(21) since it is not positive for all x .

We conclude that every positive solution of Eq. (14) is periodic with period two.

Example 2 : (Camouzis & Ladas 2008). By considering the following second-order rational difference equation

$$y(x+1)=\frac{1+y(x)}{y(x-1)}, \quad x=0,1,... \tag{22}$$

Show that every positive solution is periodic with period 5.

By using Theorem 1 and 3, the differential transform of Eq. (22) will be deduced as following:

After applying DTM on $z(x)=y(x-1)$ we obtain,

$$Z(k)=\sum_{h=k}^{N}\binom{h}{k}(-1)^{h-k}Y(h)$$

Therefore,

$$Z(0)=\sum_{h=0}^{N}\binom{h}{0}(-1)^{h}Y(h)=\sum_{h=0}^{N}(-1)^{h}Y(h)$$

and

$$Z(k-\ell)=\sum_{h=k}^{N}\binom{h}{k-\ell}(-1)^{h-(k-\ell)}Y(h)$$

Thus, after applying DTM on Eq.(22) by using Eq.(15) we have

$$\sum_{h=k}^{N}\binom{h}{k}Y(h)=\frac{\delta(k)+Y(k)-\sum_{\ell=0}^{k-1}(\sum_{h=\ell}^{N}\binom{h}{\ell}Y(h))(\sum_{h=k-\ell}^{N}\binom{h}{k-\ell}(-1)^{h-(k-\ell)}Y(h))}{\sum_{h=0}^{N}(-1)^{h}Y(h)} \tag{23}$$

hence,

$$(\sum_{h=0}^{N}(-1)^{h}Y(h))(\sum_{h=k}^{N}\binom{h}{k}Y(h))+\sum_{\ell=0}^{k-1}(\sum_{h=\ell}^{N}\binom{h}{\ell}Y(h))$$
$$\times(\sum_{h=k-\ell}^{N}\binom{h}{k-\ell}(-1)^{h-(k-\ell)}Y(h))-Y(k)=\delta(k) \tag{24}$$

By taking N=3*,* the following system of equations can be obtained from Eq.(24) for k=0, 1, 2, 3.

$$(Y(0)+Y(1)+Y(2)+Y(3))(Y(0)-Y(1)+Y(2)-Y(3))-Y(0)\ =\ 1$$
$$(Y(1)+2Y(2)+3Y(3))(Y(0)-Y(1)+Y(2)-Y(3))+$$
$$+(Y(1)-2Y(2)+3Y(3))(Y(0)+Y(1)+Y(2)+Y(3))-Y(1)=0$$
$$(Y(2)+3Y(3))(Y(0)-Y(1)+Y(2)-Y(3))+$$
$$+(Y(2)-3Y(3))(Y(0)+Y(1)+Y(2)+Y(3))+ \tag{25}$$
$$+(-Y(2)+Y(1))(Y(0)+2Y(1)+3Y(2)+4Y(3))-Y(2)=0$$

we get from the two-equation system (25):

$Y(0)=1.6180340, Y(1)=0, Y(2)=0, Y(3)=0,$

$Y(0)=1.1221537, Y(1)=0.43192290E\text{-}1, Y(2)=0.33515247, Y(3)=\text{-}0.33488742E\text{-}2.$

Then by using the inverse transformation rule at $x_0=0$ in Eq.(2), the following two series solutions are obtained :

$$y_1(x)=1.6180340+O(x^4) \tag{26}$$

$$y_2(x)=1.1221537+0.43192290\text{E-}1x+0.33515247x^2\text{-}0.33488742\text{E-}2x^3+O(x^4) \tag{27}$$

As we can notice in Eq.(26), it is comprehensible that this solution is positive for all x. In addition, $y_1(x)$=1.6180340+$O(x^4)$ satisfies the condition of period 5, such that $y_1(x+5)=y_1(x)=1.6180340$ for all $x\geq -1$.

However, Eq. (27) is not positive solution for every $x\geq -1$. Therefore, we do not consider this solution.

As a consequences, every positive solution which is $y_1(x)$=1.6180340+$O(x^4)$ of Eq.(22) is a periodic with period five.

Example 3: (Kulenovic` & Ladas 2002). Show that every positive solution of the following third-order rational difference equation

$$y(x+1)=\frac{1+y(x-2)}{y(x)}, \qquad x=0,1,... \tag{28}$$

is periodic with period 5.

By applying the DTM on both sides, we obtain:

$$\sum_{h=k}^{N}\binom{h}{k}Y(h)=\frac{\delta(k)+\sum_{h=k}^{N}\binom{h}{k}(-2)^{h-k}Y(h)-\sum_{\ell=0}^{k-1}(\sum_{h=\ell}^{N}\binom{h}{\ell}Y(h))Y(k-\ell)}{Y(0)}, \tag{29}$$

After we simplify Eq.(29), we get the following:

$$Y(0)\sum_{h=k}^{N}\binom{h}{k}Y(h)+\sum_{\ell=0}^{k-1}(\sum_{h=\ell}^{N}\binom{h}{\ell}Y(h))Y(k-\ell)-\sum_{h=k}^{N}\binom{h}{k}(-2)^{h-k}Y(h)=\delta(k) \tag{30}$$

By choosing N=3, the following system of equations can be obtained from (30) for k=0, 1, 2, 3.

$$Y(0)(Y(0)+Y(1)+Y(2)+Y(3))-Y(0)+2Y(1)-4Y(2)+8Y(3)\ =\ 1$$

$$Y(0)(Y(1)+2Y(2)+3Y(3))+(Y(0)+Y(1)+Y(2)+Y(3))Y(1)-Y(1)+4Y(2)-12Y(3)\ =\ 0$$

$$(Y(0)(Y(2)+3Y(3))+(Y(0)+Y(1)+Y(2)+Y(3))Y(2)+ \tag{31}$$

$$+(Y(0)+2Y(1)+3Y(2)+4Y(3))Y(1)-Y(2)+6Y(3)=0)$$

$$Y(0)Y(3)+(Y(0)+Y(1)+Y(2)+Y(3))Y(3)+(Y(0)+2Y(1)+3Y(2)+4Y(3))Y(2)+$$

$$+(Y(0)+2Y(1)+4Y(2)+7Y(3))Y(1)-Y(3)\ =\ 0$$

We get from the equation system (31):

$Y(0) = 1.6180340,\ Y(1) = 0,\ Y(2) = 0,\ Y(3) = 0$

$Y(0)=-15.27455, Y(1)=73.44084808, Y(2)=-274.85304, Y(3)=628.267196$

Then using the inverse transformation rule at $x_0=0$ in Eq.(2), the following two series solutions are obtained:

$$y_1(x)=1.6180340+O(x^4) \tag{32}$$

$$y_2(x)=-15.27455+73.44084808x-274.85304x^2+628.267196x^3+O(x^4) \tag{33}$$

As we can see in Eq.(32) it is obvious that this solution is positive for all *x*. In addition, Eq. (32) satisfies the condition of period five, such that, $y_1(x+5)=y_1(x)=1.6180340$ for all $x\geq -1$.

Conversely, Eq. (33) is not a positive solution to every $x\geq -1$.Therefore, the series solution in Eq. (33) of Eq.(28) is ignored.

Hence, every positive solution of Eq. (28) is periodic with period five.

Example 4 :(Grove & Ladas 2005). Show that every positive solution of the following third-order rational difference equation which is called *Todd's equation*

$$y(x+1)=\frac{1+y(x)+y(x-1)}{y(x-2)} \qquad x=0,1,.... \tag{34}$$

is periodic with period-8.

Using the same former steps by using Theorems (1) and (2):

The transformed equation of Eq.(34) by applying the DTM as follows:

$$\sum_{h=k}^{N}\binom{h}{k}Y(h)=$$

$$\frac{\delta(k)+Y(k)+\sum_{h=k}^{N}\binom{h}{k}(-1)^{h-k}Y(h)-\sum_{\ell=0}^{k-1}(\sum_{h=\ell}^{N}\binom{h}{\ell}Y(h))(\sum_{h=k-\ell}^{N}\binom{h}{k-\ell}(-2)^{h-(k-\ell)}Y(h))}{\sum_{h=0}^{N}(-2)^{h}Y(h)} \tag{35}$$

By multiplying both sides of Eq.(35) by $\sum_{h=0}^{N}(-2)^{h}Y(h)$ we have

$$\sum_{h=0}^{N}(-2)^{h}Y(h).\sum_{h=k}^{N}\binom{h}{k}Y(h)=\delta(k)+Y(k)+$$

$$+\sum_{h=k}^{N}\binom{h}{k}(-1)^{h-k}Y(h)-\sum_{\ell=0}^{k-1}(\sum_{h=\ell}^{N}\binom{h}{\ell}Y(h))(\sum_{h=k-\ell}^{N}\binom{h}{k-\ell}(-2)^{h-(k-\ell)}Y(h)) \tag{36}$$

If we take for example *N*=4 and *k* = 0,1,2,3,4 we obtain the following system of equations>

$$(Y(0)-2Y(1)+4Y(2)-8Y(3)+16Y(4))(Y(0)+Y(1)+Y(2)+Y(3)+Y(4))=$$
$$=1+2Y(0)-Y(1)+Y(2)-Y(3)+Y(4),$$

$$(Y(0)-2Y(1)+4Y(2)-8Y(3)+16Y(4))(Y(1)+2Y(2)+3Y(3)+4Y(4))=$$
$$=2Y(1)-2Y(2)+3Y(3)-4Y(4)-(Y(1)-4Y(2)+$$
$$+12Y(3)-32Y(4))(Y(0)+Y(1)+Y(2)+Y(3)+Y(4)),$$

$$(Y(0)-2Y(1)+4Y(2)-8Y(3)+16Y(4))(Y(2)+3Y(3)+6Y(4))=$$
$$=2Y(2)-3Y(3)+6Y(4)-(Y(2)-6Y(3)+24Y(4))(Y(0)+Y(1)+Y(2)+Y(3)+Y(4))+$$
$$-(-3Y(2)+6Y(3)-8Y(4)+Y(1))(Y(0)+2Y(1)+3Y(2)+4Y(3)+5Y(4)), \quad (37)$$

$$(Y(0)-2Y(1)+4Y(2)-8Y(3)+16Y(4))(Y(3)+4Y(4))=$$
$$2Y(3)-4Y(4)-(Y(3)-8Y(4))(Y(0)+Y(1)+Y(2)+Y(3)+Y(4))+$$
$$-(-5Y(3)+16Y(4)+Y(2))(Y(0)+2Y(1)+3Y(2)+4Y(3)+5Y(4))+$$
$$-(7Y(3)-16Y(4)-3Y(2)+Y(1))(Y(0)+2Y(1)+4Y(2)+7Y(3)+11Y(4))$$

and

$$(Y(0)-2Y(1)+4Y(2)-8Y(3)+16Y(4))Y(4)=$$
$$=2Y(4)-Y(4)(Y(0)+Y(1)+Y(2)+Y(3)+Y(4))-$$
$$+(-7\ Y(4)+Y(3))(Y(0)+2Y(1)+3Y(2)+4Y(3)+5Y(4))+$$
$$-(17Y(4)-5Y(3)+Y(2))(Y(0)+2Y(1)+4Y(2)+7Y(3)+11Y(4))+$$
$$-(-15Y(4)+7\ Y(3)-3\ Y(2)+Y(1))\ (Y(0)+2Y(1)+4Y(2)+8Y(3)+15Y(4)).$$

By using Maple we can solve the equations system in Eq. (37) and the following two solutions are obtained:

$Y(0)=2.4142136,\ \ Y(1)=0,\ \ Y(2)=0,\ \ Y(3)=0$ and $Y(4)=0.$

$Y(0)=0.48502597,\ \ Y(1)=-0.21135210,\ \ Y(2)=-0.76411913,\ \ Y(3)=0\ .79988641,$ and $Y\ (4)=-0.24657745$

Then using the inverse transformation rule at $x_0=0$ in Eq. (2), the following two series solutions are obtained:

$$y_1(x)=2.4142136+O(x^5) \quad (38)$$

$$y_2(x)=0.48502597-0.21135210x-0.76411913x^2+$$
$$+0.79988641x^3-0.24657745x^4+O(x^5) \quad (39)$$

It is clear the Eq. (38) is a positive solution of the rational difference equation in Eq. (39) .

Now we examine whether this solution is periodic with period 8 or not. Since Eq. (38) is constant function we consider this solution as a periodic with period eight such that $y_1(x+8)=y_1(x)=2.4142136+O(x^5)$ for all $x=0,1,...$

In contrast, the second solution which is in Eq. (39) is a negative solution for some *x*. Accordingly, this solution is disregarded.

We conclude from the earlier discussion that every positive solution of Eq.(34) is periodic with period eight.

Now we will consider another way to solve the rational difference equation. We have used the direct way which is the rule in Eq. (14). Now we will try to solve rational difference equations by using the previous methods and theorems.

Example 5 : Consider the previous Example in Eq.(22)

$$y(x+1)=\frac{1+y(x)}{y(x-1)}, \quad x=0,1,...$$

We can get the following after cross-multiplication:

$$y(x+1)y(x-1)-y(x)=1 \tag{40}$$

By applying the DTM on the both sides of Eq. (40), we obtain the following:

$$\sum_{k_1=0}^{k}\sum_{h_1=k_1}^{N}\sum_{h_2=k-k_1}^{N}\binom{h_1}{k_1}\binom{h_2}{k-k_1}(-1)^{h_2-k+k_1}Y(h_1)Y(h_2)-Y(k)=\delta(k) \tag{41}$$

By choosing N=3, the next system of equations can be obtained from Eq. (41) for k = 0, 1, 2, 3.

$$\begin{aligned}
&4Y(0)^2-Y(0)Y(1)+6Y(0)Y(2)-3Y(0)Y(3)-3Y(1)^2+Y(1)Y(2)+\\
&-4Y(1)Y(3)+2Y(2)^2-Y(2)Y(3)-Y(3)^2-Y(0)=1\\
&14Y(0)Y(1)-18Y(0)Y(2)+36Y(0)Y(3)+6Y(1)^2+\\
&\qquad -12Y(1)Y(2)+26Y(1)Y(3)-10Y(2)^2+10Y(2)Y(3)+12Y(3)^2-Y(1)=0\\
&22Y(0)Y(2)-57Y(0)Y(3)+11Y(1)Y(2)-\\
&\qquad +18Y(1)Y(3)-8Y(2)^2-18Y(2)Y(3)+8Y(1)^2-Y(2)=0\\
&31Y(0)Y(3)-22Y(2)Y(3)-28Y(3)^2+19Y(1)Y(2)+12Y(2)^2-8Y(1)Y(3)-Y(3)=0
\end{aligned} \tag{42}$$

we get the following from the equations system in Eq. (42):

$Y(0)=0.640388,\ Y(1)=0,\ Y(2)=0$ and $Y(3)=0$,

$Y(0)=$ -600, $Y(1)=1000,\ Y(2)=1.44775$ and $Y(3)=$ -700.

Then by using the inverse transformation rule at $x_0=0$ in Eq.(2), the following two series solutions are obtained:

$$y_1(x)=0.640388+O(x^4) \tag{43}$$

$$y_2(x) = -600 + 1000x + 1.44775x^2 - 700x^3 + O(x^4) \tag{44}$$

For the first solution $y_1(x) = 0.640388 + O(x^4)$, it is noticeable that Eq.(43) is positive and periodic solution with period five.

However, for the second solution in Eq.(44), it is negative for some *x*. Therefore, this solution is disregarded.

As we can conclude from the former discussion, that every positive solution of the rational difference equation in Eq. (40) is a periodic and with period five.

5. Summary

Briefly, the Differential Transform Method is extended to solve difference equations of any kind and order. Rational difference equations with order 1, 2 and 3 are solved by applying the DTM which have not been explored before. The numerical results have been reported by using Maple. To sum up, it has been verified that the DTM is a reliable technique to solve rational difference equations.

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