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Solution of a problem of linear plane elasticity in region between a circular boundary with slot by boundary integrals

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Abstract. A boundary Fourier expansion method is used to solve the system of field equations of plane, linear elasticity in stresses for homogeneous, isotropic media occupying a doubly-connected domain under given pressures on the boundaries. The case is considered: A circular domain with rectangular slot. The boundary values of the relevant harmonic functions are obtained and the error in satisfying the boundary conditions is given. Comparison is carried out between the present results on the boundary and those obtained by the usual boundary collocation method. The stress function and the displacement are calculated inside the domain of the normal cross-section.

The drawbacks for each method are put in evidence. The obtained results show that the presently used method performs better than BCM for the considered type of domains, and it is thus recommended for use for the evaluation of stresses inside long tubes with cavities.

Keywords: Plane elasticity; doubly-connected domain; isotropic medium; boundary integral method.

1 Introduction

The boundary-value problems of plane elasticity for isotropic media have a wide range of applications. They are usually considered as useful approximations to the more realistic three-dimensional problems. When the domain of the solution has complicated geometry, analytical methods become inefficient. The numerical methods stand on the other extreme, but their main disadvantage is that they do not produce formulae for the solution and large computational capabilities are also usually necessary, in addition to the problems raised by the stability of the numerical scheme. In the past few decades, the semi-analytical methods, in combination with the boundary techniques, have gained more popularity as being efficient and require less computational effort than the numerical approaches. Moreover, they produce approximate formulae for the solution and the resulting error can be easily evaluated in many circumstances. Trefftz's method is no doubt the most familiar boundary technique. It requires expansion of the solution in a properly chosen base, then to determine the expansion coefficients using the boundary values of the unknown function [16]. Different aspects of this theory related to the completeness property of the used expansion basis and others were considered in[9], [10], [19], [11]. An overview of the method may be found in [12]. When the satisfaction of the boundary conditions is carried out pointwise, this gives rise to the well-known Boundary Collocation Method (BCM). An extensive literature exists on the use of this method, among which we cite [12], [13], [1]. When the basis functions are taken as logarithms of the distance with origins lying outside the domain of solution, this is the Method of Fundamental Solutions treated by many authors [7], [14]. An application for doubly-connected regions is carried out in [6].

In the present work, we solve the generalized, plane Lame'problem in linear, isotropic elasticity for an infinite hollow cylinder subjected to constant pressures on its lateral surfaces. The case will
be considered, for which the normal cross-section is bounded either by a circle and a slot (rectangle).

We calculate the boundary values of the two basic harmonic functions through which the solution of the problem is determined in two ways, BCM and BFEM. The error in satisfying the boundary conditions is given. Two displacement components are then calculated.

2 Problem formulation

We consider an infinite hollow cylinder of an isotropic elastic medium. Let D be the normal cross-section of the cylinder. This is a two-dimensional, doubly connected region bounded by two contours C_1 and C_2 with parametric representations

$$x_1 = x_1(\theta) \quad \& \quad y_1 = y_1(\theta), \qquad (1)$$

$$x_2 = x_2(\theta) \quad \& \quad y_2 = y_2(\theta), \qquad (2)$$

where θ is the angular parameter measured, as usual, counter-clockwise from the x-axis of a system of Cartesian coordinates (x, y, z) with center O in the cavity and z -axis along the generators of the cylinder.

The cylinder is acted upon by pressures $p_1(\theta)$ and $p_2(\theta)$ on the lateral surfaces. Thus the considered problem is a generalized Lamé problem.

It is required to find the stresses and the displacement at all points of the cross-section D .

The basic equations and boundary conditions of the two-dimensional theory of elasticity may be found in standard textbooks. Here, we give a brief presentation of these equations along the guidelines given by Abou-Dina and Ghaleb [2], [3]. and based on previous work by A.S. Deeb, Entesar Omar Alarabi and A.O.El-Refaie [4].

Let τ_1 and $\mathbf{n}_1$, τ_2 and $\mathbf{n}_2$ denote respectively the unit vectors tangent and normal to C_1 and C_2 at arbitrary points, the positive sense associated with C_1 and C_2 being taken in the counter-clockwise sense. One has

$$\tau_1 = \frac{\dot{x}_1}{\omega_1} i + \frac{\dot{y}_1}{\omega_1} j \quad \& \quad \mathbf{n}_1 = \frac{\dot{y}_1}{\omega_1} i - \frac{\dot{x}_1}{\omega_1} j, \qquad (3)$$

$$\tau_2 = \frac{\dot{x}_2}{\omega_2} i + \frac{\dot{y}_2}{\omega_2} j \quad \& \quad \mathbf{n}_2 = \frac{\dot{y}_2}{\omega_2} i - \frac{\dot{x}_2}{\omega_2} j, \qquad (4)$$

where the dot over a symbol denotes differentiation with respect to the parameter θ, and

$$\omega_1 = \sqrt{\dot{x}_1^2 + \dot{y}_1^2}, \qquad \omega_2 = \sqrt{\dot{x}_2^2 + \dot{y}_2^2}. \qquad (5)$$

In case the contour parameter is the arc length, the corresponding value of ω is unity. Clearly, the contours C_1 and C_2 should belong, at least, to the class C^1 so as to uniquely define the above defined unit vectors at each point.

3 Basic equations

In this section, the well-known basic equations governing the plane theory of linear elasticity are presented in accordance with [2], the representation of harmonic functions is briefly discussed.

3.1 Field equations

In the absence of body forces, the stress tensor components in the plane may be expressed by means of one single auxiliary function, called the stress function or Airy's function, subsequently denoted U . In fact, the equations of equilibrium

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}=0,$$

$$\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}=0. \qquad (6)$$

are automatically satisfied if the identically non-vanishing stress components are defined through the function U by the relations:

$$\sigma_{xx}=\frac{\partial^2 U}{\partial y^2},\qquad \sigma_{yy}=\frac{\partial^2 U}{\partial x^2},\qquad \sigma_{xy}=-\frac{\partial^2 U}{\partial x\,\partial y}. \qquad (7)$$

It is well-known that the biharmonic function may be expressed in terms of two harmonic functions according to the representation

$$U=x\,\phi+y\,\phi^c+\psi, \qquad (8)$$

where "c" denotes the harmonic conjugate. Thus, the stress components may be rewritten in terms of the harmonic functions as:

$$\sigma_{xx}=x\,\frac{\partial^2\phi}{\partial y^2}+2\frac{\partial\phi^2}{\partial y}+y\,\frac{\partial^2\phi^c}{\partial y^2}+\frac{\partial^2\psi}{\partial y^2},$$

$$\sigma_{xy}=-x\,\frac{\partial^2\phi}{\partial x\,\partial y}-y\,\frac{\partial^2\psi^2}{\partial x\,\partial y}-\frac{\partial^2\psi}{\partial x\,\partial y},$$

$$\sigma_{yy}=x\,\frac{\partial^2\phi}{\partial x^2}+2\frac{\partial\phi}{\partial x}+y\,\frac{\partial^2\phi^c}{\partial x^2}+\frac{\partial^2\psi}{\partial x^2} \qquad (9)$$

The generalized Hooke's law reads

$$\sigma_{xx}=\frac{\nu E}{(1+\nu)(1-2\nu)}(\frac{\partial u}{\partial x}+\frac{\partial\upsilon}{\partial y})+\frac{E}{1+\nu}\frac{\partial u}{\partial x},$$

$$\sigma_{xy}=\frac{E}{2(1+\nu)}(\frac{\partial u}{\partial y}+\frac{\upsilon}{\partial x}),$$

$$\sigma_{yy}=\frac{\nu E}{(1+\nu)(1-2\nu)}(\frac{\partial u}{\partial x}+\frac{\partial\upsilon}{\partial y})+\frac{E}{1+\nu}\frac{\partial\upsilon}{\partial y}, \qquad (10)$$

where E and ν denote Young's modulus and Poisson's respectively. Using the above relations together with (4), one arrives at:

$$\frac{E}{1+\nu}u=-\frac{\partial U}{\partial x}+4(1-\nu)\phi,$$

$$\frac{E}{1+\nu}\upsilon=-\frac{\partial U}{\partial y}+4(1-\nu)\phi^c, \qquad (11)$$

which may be rewritten as:

$$2\mu u = (3-4\nu)\phi - x\frac{\partial\phi}{\partial x} - y\frac{\partial\phi^c}{\partial x} - \frac{\partial\psi}{\partial x}, \qquad (12)$$

$$2\mu\upsilon = (3-4\nu)\phi^c - x\frac{\partial\phi}{\partial y} - y\frac{\partial\phi^c}{\partial y} - \frac{\partial\psi}{\partial y}, \qquad (13)$$

where $\mu = \frac{E}{2(1+\nu)}$ denotes the shear modulus.

$$\begin{aligned}
\phi(x,y) &= a_o x + b_o y + c_o xy + d_o(y^2 - x^2) \\
&+ \sum_{n=1}^{N}(a_n \cos nx \cosh ny + b_n \cos nx \sinh ny \\
&+ c_n \sin nx \cosh ny + d_n \sin nx \sinh ny) + A, \qquad (14)
\end{aligned}$$

$$\begin{aligned}
\phi^c(x,y) &= a_o y - b_o x + \frac{1}{2}c_o(y^2 - x^2) - 2d_o xy \\
&+ \sum_{n=1}^{N}(-a_n \sin nx \sinh ny - b_n \sin nx \cosh ny \\
&+ c_n \cos nx \sinh ny + d_n \cos nx \cosh ny) + B, \qquad (15)
\end{aligned}$$

$$\begin{aligned}
\psi(x,y) &= f_o x + g_o y + h_o xy + k_o(y^2 - x^2) \\
&+ \sum_{n=1}^{N}(f_n \cos nx \cosh ny + g_n \cos nx \sinh ny \\
&+ h_n \sin nx \cosh ny + k_n \sin nx \sinh ny) + C. \qquad (16)
\end{aligned}$$

$$U = x\phi + y\phi^c + \psi \qquad (17)$$

$$\begin{aligned}
U &= a_o(x^2 + y^2) + \frac{1}{2}c_o(x^2 + y^2) - d_o(x^2 + y^2)x \\
&+ \sum_{n=1}^{N} x(a_n \cos nx \cosh ny + b_n \cos nx \sinh ny \\
&+ c_n \sin nx \cosh ny + d_n \sin nx \sinh ny) \\
&+ \sum_{n=1}^{N} y(-a_n \sin nx \sinh ny - b_n \sin nx \cosh ny \\
&+ c_n \cos nx \sinh ny + d_n \cos nx \cosh ny) \\
&+ f_o x + g_o y + h_o xy + k_o(y^2 - x^2) \\
&+ \sum_{n=1}^{N}(f_n \cos nx \cosh ny + g_n \cos nx \sinh ny \\
&+ h_n \sin nx \cosh ny + k_n \sin nx \sinh ny) + Ax + By + G. \qquad (18)
\end{aligned}$$

$$\sigma_{nn} = (\sigma_{xx} n_x + \sigma_{xy} n_y) n_x + (\sigma_{xy} n_x + \sigma_{yy} n_y) n_y , \qquad (19)$$

$$\sigma_{n\tau} = -(\sigma_{xx} n_x + \sigma_{xy} n_y) n_y + (\sigma_{xy} n_x + \sigma_{yy} n_y) n_x . \qquad (20)$$

$$\begin{aligned}
&\sigma_{nn} = n_x^2 (2a_o + 3c_o y - 2d_o x + 2k_o) \\
&+(n_x^2 - n_y^2)\Big(\sum_{n=1}^{N} x\,(n^2 a_n \cos nx \cosh ny + n^2 b_n \cos nx \sinh ny \\
&+n^2 c_n \sin nx \cosh ny + n^2 d_n \sin nx \sinh ny) \\
&+\sum_{n=1}^{N} y\,(-n^2 a_n \sin nx \sinh ny - n^2 b_n \sin nx \cosh ny \\
&+n^2 c_n \cos nx \sinh ny + n^2 d_n \cos nx \cosh ny)\Big) \\
&+(n_x^2 + n_y^2)\Big(\sum_{n=1}^{N} 2(-n a_n \sin nx \cosh ny - n b_n \sin nx \sinh ny \\
&+n c_n \cos nx \cosh ny + n d_n \cos nx \sinh ny)\Big) \\
&+(n_x^2 - n_y^2)\Big(\sum_{n=1}^{N} (n^2 f_n \cos nx \cosh ny + n^2 g_n \cos nx \sinh ny \\
&+n^2 h_n \sin nx \cosh ny + n^2 k_n \sin nx \sinh ny)\Big) \\
&+n_y^2 (2a_o + c_o y - 6d_o x - 2k_o) + 2n_x n_y (2d_o y - h_o - c_o x) \\
&+2n_x n_y \Big(-\sum_{n=1}^{N} x\,(-n^2 a_n \sin nx \sinh ny - n^2 b_n \sin nx \cosh ny \\
&+n^2 c_n \cos nx \sinh ny + n^2 d_n \cos nx \cosh ny) \\
&-\sum_{n=1}^{N} y\,(-n^2 a_n \cos nx \cosh ny - n^2 b_n \cos nx \sinh ny \\
&-n^2 c_n \sin nx \cosh ny - n^2 d_n \sin nx \sinh ny) \\
&-\sum_{n=1}^{N} (-n^2 f_n \sin nx \sinh ny - n^2 g_n \sin nx \cosh ny \\
&+n^2 h_n \cos nx \sinh ny + n^2 k_n \cos nx \cosh ny)\Big). \qquad (21)
\end{aligned}$$

$$\begin{aligned}
&\sigma_{n\tau} = n_x n_y (-2c_o y - 4d_o x - 4k_o) \\
&+n_x n_y \Big(\sum_{n=1}^{N} 2x\,(-n^2 a_n \cos nx \cosh ny - n^2 b_n \cos nx \sinh ny \\
&-n^2 c_n \sin nx \cosh ny - n^2 d_n \sin nx \sinh ny) \\
&+\sum_{n=1}^{N} 2y\,(n^2 a_n \sin nx \sinh ny + n^2 b_n \sin nx \cosh ny \\
&-n^2 c_n \cos nx \sinh ny - n^2 d_n \cos nx \cosh ny) \\
&+\sum_{n=1}^{N} 2(-n^2 f_n \cos nx \cosh ny - n^2 g_n \cos nx \sinh ny \\
&-n^2 h_n \sin nx \cosh ny - n^2 k_n \sin nx \sinh ny)\Big)
\end{aligned}$$

$$
\begin{aligned}
&+(n_x^2-n_y^2)(2d_o y-h_o-c_o x)\\
&+(n_x^2+n_y^2)(-\sum_{n=1}^{N} x\,(-n^2 a_n \sin nx \sinh ny - n^2 b_n \sin nx \cosh ny\\
&+n^2 c_n \cos nx \sinh ny + n^2 d_n \cos nx \cosh ny)\\
&-\sum_{n=1}^{N} y\,(-n^2 a_n \cos nx \cosh ny - n^2 b_n \cos nx \sinh ny\\
&-n^2 c_n \sin nx \cosh ny - n^2 d_n \sin nx \sinh ny)\\
&-\sum_{n=1}^{N}(-n^2 f_n \sin nx \sinh ny - n^2 g_n \sin nx \cosh ny\\
&+n^2 h_n \cos nx \sinh ny + n^2 k_n \cos nx \cosh ny)). \qquad (22)
\end{aligned}
$$

4 The methods of solution

We present herebelow two boundary methods for the solution of boundary-value problems for differential equations, which will be used throughout the thesis. These two methods are in fact two variants of the well-known Trefftz's method (TM). Both methods use an expansion of the solution in a set of basis functions which satisfy the given differential equation. The remaining task is thus reduced to satisfying the boundary conditions imposed on the solution.

The first method is currently known as the Boundary Collocation Method (BCM); it relies on satisfying the boundary conditions pointwise at a number of properly chosen boundary points, called the " nodes". The second method is simply an L^2- version of the first one. Following Abou-Dina and Ghaleb [5], this second method will be called Boundary Fourier Expansion Method (BFEM).

4.1 Short presentation of the methods

Let D be a simply-connected region in the plane, bounded by a contour C of finite length L and let $t \in [0,T]$ be a real parameter characterizing the points of the contour C, starting from a point P_0 on C. In particular, t may be the arc length s measured on C anticlockwise as usual, starting from P_0. Extension to doubly-connected domains, the case of present interest, is straightforward.

Consider the following boundary-value problem for the partial differential equation in the unknown function U :

$$K(U(\mathbf{r}))=0 \qquad \text{in} D, \qquad (23)$$

$$W\,U(t)=f(t) \qquad \text{on} C, \qquad (24)$$

where $\mathbf{r}$ is the position vector of a general point $P \in D$, K and W are linear partial differential operators and f is a given function on C. Special cases of this problem may be the Dirichlet's, the Neumann's and the mixed boundary-value problems. The case of multiple differential equations and boundary conditions is a straightforward generalization.

Consider now a complete set of linearly independent functions, called the " trial functions" , $\{\varphi_i(\mathbf{r}), i=0,1,2,...,N\}$. This set of " trial functions" is required to generate the approximate solution $U_a(\mathbf{r})$ as a linear combination of the functions $\varphi_i(\mathbf{r})$ with a certain error tolerance. One such set used for Laplace's equation is the well-known set of Cartesian harmonics

$$\{1,\cos(nx)\cosh(ny),\cos(nx)\sinh(ny),\sin(nx)\cosh(ny),\sin(nx)\sinh(ny), \qquad n=1,2,...\}$$

in which we are presently interested.

An additional factor determining the choice of the trial functions would be the possibility of satisfaction of some boundary condition on certain parts of the boundary from the outset. Thus, the linear combination

$$U_a(\mathbf{r}) = \sum_{i=0}^{N} a_i \varphi_i(\mathbf{r}) \qquad (25)$$

rigorously satisfies equation (4.23) and, possibly, the boundary condition (4.24) on certain parts of the boundary. The number N is usually referred to as the " number of degrees of freedom ". The unknown coefficients $\{a_i, i=0,1,2...,N\}$ will now be determined so as to enforce the boundary condition on the remaining part of the boundary.

In the BCM, this is simply achieved by satisfying (4.24) at a certain number M of boundary points, called the " collocation points". This is the most direct way of getting an approximate solution to problem $(4.23)+(4.24)$. One inconvenience of this method, however, is the arbitrariness in choosing the number and the location of the boundary points at which the boundary condition is enforced. Also, if M is increased beyond a certain limit, a question of crowdedness of the boundary points may arise, that render the numerical analysis more delicate, if not impossible at all, due to rounding-off and accumulation errors and to instability. This approach is usually implemented by the use of optimization techniques, generally nonlinear, a fact that drastically increases the solution cost. The method ultimately leads to a rectangular system of linear algebraic equations for the coefficients a_i.

Enforcing the boundary condition may also be achieved, not pointwise, but in " the mean". This leads to approximations of the solution in the sense of L^2-space, as for the standard techniques based on Variational Principles. The resulting set of linear algebraic equations in this case is square, but the matrix elements are now expressed as integrals that need, in general, to be evaluated numerically.

The method proposed hereafter (BFEM) may be considered as a variant of the standard method of approximation of the solution " in the mean" . It generally leads to rectangular systems of linear equations and to integrals that are simpler to evaluate than in the standard method and relies on the following idea: Substitution of (1.3) into (1.2) yields the " error in satisfying the boundary condition" on C :

$$ER(t) \equiv \sum_{n=0}^{N} a_n W \varphi_n(t) - f(t), \qquad t \in [0,T]. \qquad (26)$$

Extending the function $ER(t)$ evenly to the interval $[-T,0]$, one obtains a function that, hopefully, should vanish on $[-T,T]$. The Fourier coefficients of this function with respect to the orthonormal set of functions $\{1, \cos\frac{m\pi t}{T}, m=1,2,...\}$ should then vanish. Setting to zero the first M Fourier coefficients generates a rectangular system of linear algebraic equations of size $M \times N$ for the expansion coefficients

$\{a_i, i = 0,1,2...,N\}$ in the form

$$\sum_{n=0}^{N-1} A_{mn} a_n = B_m, \qquad m = 0,1,2,...,M-1, \tag{27}$$

with

$$A_{mn} = \int_0^T W\varphi_n(t)\cos\frac{m\pi t}{T}dt, \quad B_m = \int_0^T f(t)\cos\frac{m\pi t}{T}dt. \tag{28}$$

It may also happen that we do not extend the function $ER(t)$ evenly as explained above, in which case we have to consider all the other Fourier coefficients involving *sines* as well.

The resulting systems of linear algebraic equations will be solved using the well-known method of " Least Squares". The number M may be increased until some error criterion is satisfied. For our purposes, one of two measures of error will be considered hereafter:

1. the maximal boundary error (ERB) measuring the largest error in satisfying the boundary conditions:

$$ERB = \sup_{t\in 0,T\,]} |ER(t)|, \tag{29}$$

2. the maximal solution error (ERS) measuring the largest error between the approximate solution $U_a(\mathbf{r})$ and the exact solution (assumed known) $U_e(\mathbf{r})$ at a certain properly chosen set of points in the domain of the solution:

$$ERS = \max_k |U_a(\mathbf{r}_k) - U_e(\mathbf{r}_k)|. \tag{30}$$

When the problem under consideration is a Dirichlet's problem, then ERB will be used, since the maximum error in the solution is expected to be reached at the boundary.

For more complicated cases, where there is more than one boundary condition, the same technique may be used invariably. For this, one has only to link additional intervals to $[-T,T]$ corresponding to the additional boundary conditions. This will indeed be the case of the considered problems, when the domain of the solution is doubly-connected and, consequently, there are two boundary conditions to be addressed. Here,

5 Numerical results

In the subsequent figures, the dashed curves calculated on the boundary are obtained by BCM, while the plain curves are obtained by BFEM.

5.1 The circular cylinder with rectangular slot

Consider an infinitely long tube of an isotropic elastic medium, with normal cross-section bounded by two contours. Let D be the normal cross-section of the tube. This is a two-dimensional, doubly connected region bounded by circle boundary C1 and rectangle slot C2 with parametric representations

$$x_1(\theta) = a_1\cos\theta, \qquad y_1(\theta) = a_1\sin\theta.$$

and

$$x_2(s)=\begin{cases} -a_2, & -2a_2-2b_2 \le s \le -2a_2-b_2 \\ s+b_2+a_2, & -2a_2-b_2 \le s \le -b_2 \\ a_2, & -b_2 \le s \le b_2 \\ -s+a_2+b_2, & b_2 \le s \le 2a_2+b_2 \\ -a_2, & 2a_2+b_2 \le s \le 2a_2+2b_2 \end{cases}$$

$$y_2(s)=\begin{cases} s-2a_2-2b_2, & -2a_2-2b_2 \le s \le -2a_2-b_2 \\ -b_2, & -2a_2-b_2 \le s \le -b_2 \\ s, & -b_2 \le s \le b_2 \\ b_2, & b_2 \le s \le 2a_2+b_2 \\ -s+2a_2+2b_2, & 2a_2+b_2 \le s \le 2a_2+2b_2 \end{cases}$$

We take that pressures p_1, p_2 are specified on the two boundaries $C1$, $C2$ in the period $0<\theta\le 2\pi$, $-2a-2b<s\le 2a+2b$.

$$\sigma_{nn}=p_1, \qquad \sigma_{n\tau}=0.$$

on $C1$,

$$\sigma_{nn}=p_2, \qquad \sigma_{n\tau}=0.$$

on $C2$.

The above equations are solved numerically using Mathematica software, from which we have acquired the boundary values of the basic harmonic functions ϕ, ϕ^c ,ψ and displacements u , v . This is shown on the following figures:

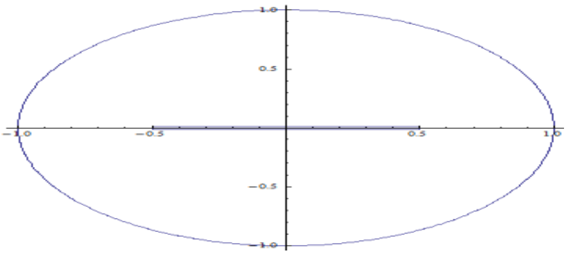

Figure 1 :Circular normal cross-sections with central rectangular slot for $a_1=1, a_2=0{,}5, b_2=0.01, p_1=1, p_2=1.$

The values of the pressures on both boundaries are

$$\sigma_{nn} = -p_1, \qquad \sigma_{n\tau} = 0 \qquad \text{on } C_1$$

$$\sigma_{nn} = -p_2, \qquad \sigma_{n\tau} = 0 \qquad \text{on } C_2$$

and the error in satisfying the boundary conditions is taken by

$$ERB = \int_0^{2\pi} \left(\overset{(1)}{\sigma}_{nn} + p_1 + \overset{(2)}{\sigma}_{nn} + p_2 \right) d\theta$$

The above equations are solved numerically, their solution provides the boundary values of the basic harmonic functions ϕ, ϕ^c, ψ, the displacements u, v. In BFEM, we used 6 terms in the summations for the different unknown functions, i.e **N = 6** in (4.27) and (4.28).

The corresponding number of zeroed Fourier coefficients was **M = 14** for each of the four boundary conditions. The maximum error resulting from the use of this method is $ERB = 1.3 \times 10^{-13}$.

Further increase of the value of **M** up to **50** kept the results almost unchanged and no instability was observed. We did not go beyond this value of M, but it is thought that there is an upper limit for **M**, after which the results begin to deteriorate. In BCM, we used **3** terms only in the summations for the different unknown functions. The corresponding number of boundary points needed for the calculations was equal to $(8 \times 3) + 5 = 29$. where the digit **5** refers to the number of unknown coefficients in the expansions, figuring outside the summation signs and entering into the expressions for the stresses. The maximum error resulting from the use of this method for this number of points is $ERB = 1.5 \times 10^{-18}$. An increase in the number of boundary points resulted in a deterioration of the results, especially for the function ψ which kept changing as the number of nodes was increased. The results are shown on the plots of figures (2). Although the errors in satisfying the boundary conditions are quite low for both BFEM and BCM, the curves obtained by the two methods do not match perfectly. We notice that the maxima of the functions as calculated by BFEM are higher than the corresponding ones found by BCM. The inverse is true for the minimal values. This discrepancy is compatible with the following three facts: First, the well-known non uniqueness of solutions of the equations of plane, linear elasticity for the functions ϕ and ψ ; second, the BFEM approximates "in the mean", while BCM provides pointwise approximation.

The function ψ is almost zero on both boundaries. In BCM, this function is also obtained constant, but different from zero, which does not reflect on the values of stresses and displacements.

However, as noted earlier, the values obtained by BCM keep changing as the number of nodal points changes. Such results are not trustworthy.

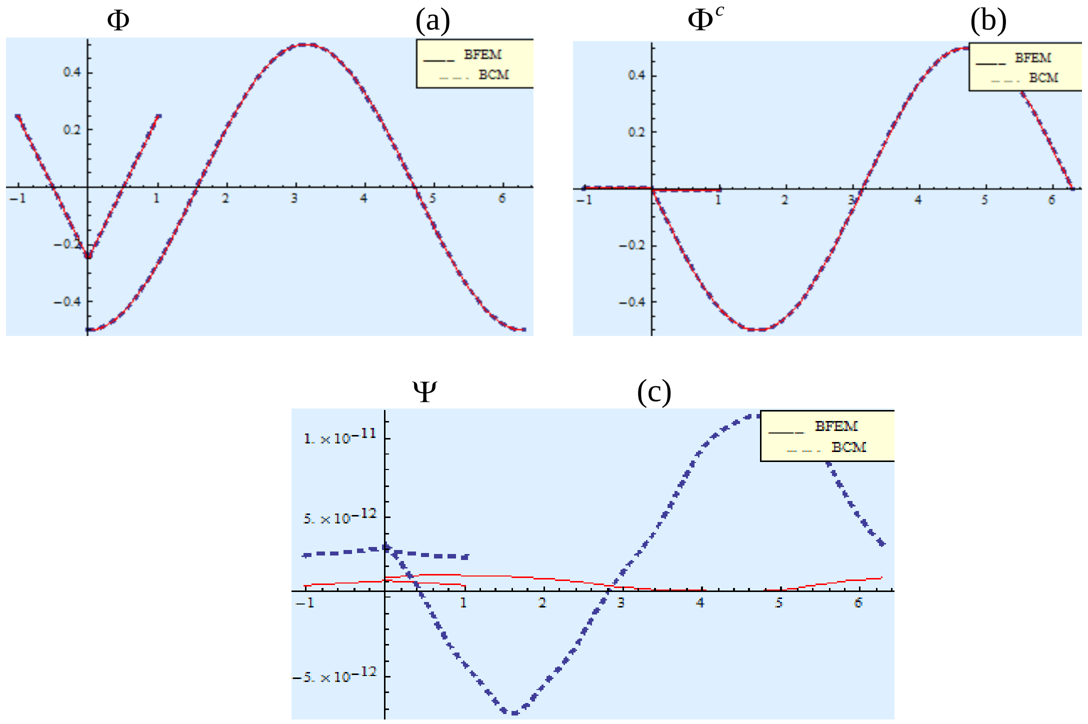

Figure 2: Harmonic functions (a) Φ; (b) Φ^c; (c) Ψ on the circular cross-section with rectangular slot.

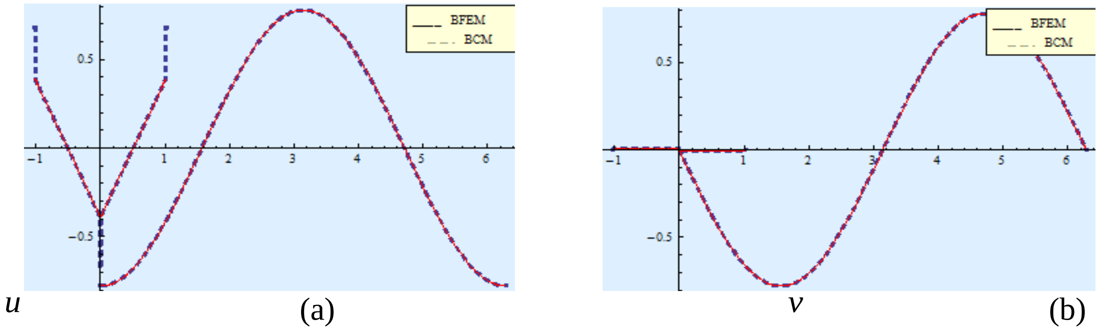

Figure 3: Displacement components (a) u; (b) v on the boundaries of circular cross-section with rectangular slot.

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