مجـلة الـتربـوي
Journal of Educational
ISSN: 2011- 421X
Arcif Q3

معامل التأثير العربي 1.5
العدد 19

مجلة التربوي
مجلة علمية محكمة تصدر عن كلية التربية
جامعة المرقب

العدد التاسع عشر
يوليو 2021م

هيئـــة تحرير
مجلة التربوي

- **المجلة ترحب بما يرد عليها من أبحاث وعلى استعداد لنشرها بعد التحكيم .**
- **المجلة تحترم كل الاحترام آراء المحكمين وتعمل بمقتضاها .**
- **كافة الآراء والأفكار المنشورة تعبر عن آراء أصحابها ولا تتحمل المجلة تبعاتها .**
- **يتحمل الباحث مسؤولية الأمانة العلمية وهو المسؤول عما ينشر له .**
- **البحوث المقدمة للنشر لا ترد لأصحابها نشرت أو لم تنشر .**

ضوابط النشر:

يشترط في البحوث العلمية المقدمة للنشر أن يراعى فيها ما يأتي :

- أصول البحث العلمي وقواعده .
- ألا تكون المادة العلمية قد سبق نشرها أو كانت جزءا من رسالة علمية .
- يرفق بالبحث تزكية لغوية وفق أنموذج معد .
- تعدل البحوث المقبولة وتصحح وفق ما يراه المحكمون .
- التزام الباحث بالضوابط التي وضعتها المجلة من عدد الصفحات ، ونوع الخط ورقمه ، والفترات الزمنية الممنوحة للتعديل ، وما يستجد من ضوابط تضعها المجلة مستقبلا .

تنبيهات :

- للمجلة الحق في تعديل البحث أو طلب تعديله أو رفضه .
- يخضع البحث في النشر لأولويات المجلة وسياستها .
- البحوث المنشورة تعبر عن وجهة نظر أصحابها ، ولا تعبر عن وجهة نظر المجلة .

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مجلة التربوي
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ISSN: 2011- 421X
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العدد 19

Hilbert Space and Applications

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Abstract: Functional analysis is one of the most important branches of modern Mathematics and Physics. In functional analysis, Hilbert spaces have a great and positive influence in the development of functional analysis. Hilbert spaces have a rich geometric structure because they are endowed with a scalar product which allows the introduction of the concept of orthogonality of vectors. The aim of this paper is to introduce the Hilbert spaces, and their properties, operations, and applications. We first give an introduction about functional analysis and highlight the importance of Hilbert spaces in the development of functional analysis. Then, we focus on significant and crucial spaces for Hilbert spaces called Banach Spaces. We do so by studying normed spaces and their properties. Finally, we discuss what is meant by Hilbert spaces and what is the relation between Hilbert and Banach spaces.

***Keywords**: Functional analysis, Hilbert Space, Banach Spaces.*

1. INTRODUCTION

Mathematics is the queen of science and the language of nature. Also, it occupies and plays a crucial and important role in the human societies. Moreover, it represents a strategic key in the development of the whole world and mankind. It has roots in ancient Egypt and Babylonia, then grew rapidly in ancient Greece. Mathematics written in ancient Greek was translated into Arabic. About the same time some mathematics of India was translated into Arabic. Later some of this mathematics was translated into Latin and became the mathematics of Western Europe and then after a hundred years it became the mathematics of the whole world. In fact, Mathematics can be found in plenty of fields today for instance Business Decisions, Physics, Economics, Engineering, Medicine, Modern Society, Industry, Finance, Marketing and Computer Science. As we see nowadays, we are in the era of computer and technology, but if we think deeply for a moment, we will find that Mathematics has played a main and significant role in the high development of Computer Science. All the latest software and programmes that made the world very small and provided comfortable life for us has been connected with some Mathematical programmes.

There are plenty of branches of Mathematics such as Geometry, Topology, Algebra, Number Theory, Mathematical Analysis including Calculus and Real Analysis, Complex Variable Analysis, Differential Equations, Numerical Analysis and Functional Analysis etc. All these branches are relevant and connected with one another. And all of them have their own effects, significances, and applications in several fields. One of the most crucial and important branches is the Functional Analysis. Functional Analysis was born in the early years of the twentieth century as part of a larger trend toward abstraction. One of the characteristics of functional analysis is the study of classes of functionals in particular, the dual space, the space of all continuous linear functionals on a vector space (see[1-8]). Functional Analysis is one of the central areas of modern mathematics and it has a strong influence on a great number of completely different fields inside and outside of mathematics such as systems engineering and atomic physics, or some topics in general Topology, Measure and Integration, Linear Algebra, Geometry, the theory of approximation or representation theory and theory of Banach and Hilbert spaces. These spaces have a great impact in the development of Functional Analysis because the whole theory of Functional Analysis deals with them. Banach space is known as complete normed linear space while the complete inner product space is called Hilbert space. The typical examples of Banach spaces are $\mathbb{R}^n$and $\mathbb{C}^n$under the following norm:

$$\|u\| = \sqrt{\sum_{j=1}^{n} |u_j|^2}$$

where $u = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$ or $\mathbb{C}^n$.
and,

$$C_b(\mathbb{R}) = \{f \in L^\infty(\mathbb{R}) : f \text{ is bounded and continuous}\}, \quad \|f\|_\infty = sup_{x\in R}|f(x)|$$

The typical examples of Hilbert space are as follows:
The space $\mathbb{C}^n$, *finite dimensional complex Euclidean space*, is a Hilbert space with an inner product expressed by:

$$< u, v > = u \,.\, v = u_1\overline{v_1} + u_2\overline{v_2} + \cdots + u_n\overline{v_n}.$$

where:

$$u = (u_1, u_2, \dots, u_n) \in \mathbb{C}^n \quad and \quad v = (v_1, v_2, \dots, v_n) \in \mathbb{C}^n.$$

The space$\mathbb{R}^n$with the following an inner product

$$< u, v > = u \,.\, v = u_1v_1 + u_2v_2 + \cdots + u_nv_n \quad where\ u, v \in \mathbb{R}^n.$$

The mathematical concept of a Hilbert space named after David Hilbert who made a very positive and great influence in the development of Functional analysis. It extends the methods of vector algebra and calculus from two and three dimensions to finite and infinite number of dimensions.

The theory of Hilbert spaces is the core around which functional analysis has developed. Hilbert spaces have a rich geometric structure because they are endowed with an inner product which allows the introduction of the concept of orthogonality of vectors. Hilbert space is truly fundamental mathematical structure which appears in wide branches of pure and applied mathematics. For instance, quantum mechanics, integral equations, linear system of equations and operator theory.

Also, Hilbert space is an abstract notion of great power and beauty which has been central to the development of mathematical analysis and forms the backdrop for many applications of analysis to science and engineering due to the existence of an inner product that determines the norm.

2. Normed Vector Spaces

2.1 Basics about Linear algebra

In order to achieve our aim of discussing the operators and the application of a Hilbert space, it is really important to know some concepts, definitions and examples of some related subjects such as vector space, vector subspace, normed space, Banach space etc. All these results will provide assistance and guide us to the main point of our research.

Definition 2.1.1 (Vector Space). A vector space X (called also *linear space*) is a set of elements (vectors) over the field F, with two operations:

(i) *A mapping* $(x,y) \to x+y$ *from* $X \times X$ *into* X *called addition*

(ii) *A mapping* $(\lambda,x) \to \lambda x$ *from* $F \times X$ *into* X *called multiplication by scalars, such that the following conditions are satisfied :*

$$x + y = y + x$$
$$(x + y) + z = x + (y + z).$$
$$\alpha(x + y) = \alpha x + \alpha y.$$
$$(\alpha + \beta)x = \alpha x + \beta x.$$
$$(\alpha\beta)x = \alpha(\beta x).$$
$$x + 0 = x.$$
$$0\ x = 0.$$
$$1\ x = x.$$
$$where\ x,y,z \in X\ and\ \alpha,\beta \in F$$

Elements of X will be called *vectors* If $F = \mathbb{R}$, then X will be called a *real vector space*, and if $F = \mathbb{C}$, then X will be called a *complex vector space.*

Definition 2.1.2 (Vector Subspace). A subset $W \subset V$ of a vector space is a *vector subspace* of V if thefollowing two axioms are satisfied:

$(\boldsymbol{i})$ *If* v, w *are vectors in* W*, then so is* $v + w$

$(\boldsymbol{ii})$ *For any scalar* $\lambda \in \mathbb{R}$*,*
if w *is any vector in* W*, then so is* hw

Vector subspace is also called linear subspace.

Definition 2.1.3 (Linear Combination). Let E be a vector space and let $x_1, x_2, \dots\dots, x_k \in E$. A vector $x \in E$ is called a linear combination of vectors $x_1, x_2, \dots, x_k$ if there exist scalars $\alpha_1, \alpha_2, \dots\dots, \alpha_k$ such that

$$x = \alpha_1 x_1 + \cdots + \alpha_k x_k$$

Definition 2.1.4 (Span). Given a set of vectors $S = \{v_i \in V : i \in I\}$ for a vector space V and indexing set I, the collection of all linear combinations of vectors in S is called the*span* of S and denoted by *span(S)*
It is obvious that *span(S)* is a subspace of V for any set S.

Definition 2.1.5 (Linear Independence).

A finite collection of vectors $\{x_1, x_2, \dots, x_k\}$ is called *linearly dependent* if $\alpha_1 x_1 + \cdots + \alpha_k x_k = 0$ only if
$\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0.$

A collection of vectors which is not linearly independent is called *linearly dependent.*

2.2 Basic concepts and definition

Definition 2.2.1 (Metric Space). Let X be a set. A *metric* on X is a function:

$$d : X \times X \longrightarrow \mathbb{R}^+, \quad (x,y) \mapsto d(x,y)$$

satisfying the following properties:

$$(\boldsymbol{M1}) \quad d(x,y) \geq 0, and\ d(x,y) = 0\ if\ and\ only\ if\ x = y$$

$$(\boldsymbol{M2}) \quad d(x,y) = d(y,x) \quad for\ all\ x,y \in X$$

$$(\boldsymbol{M3}) \quad d(x,z) \leq d(x,y) + d(y,z) \quad for\ all\ x,y,z \in X \quad (triangle\ inequality)$$

A ***metric space*** (*X*,*d*) is a non-empty set X on which a metric d is given.

Definition 2.2.2 (Norm). Assume X be a vector space over $K - (i.e. \mathbb{R}\ or\ \mathbb{C})$.A *norm* on X is a map

$$\|.\| \ : \ X \ \longrightarrow \ [0,\infty[\,, \quad x \mapsto \|x\|$$

Satisfying the following properties:

$$(\boldsymbol{S2}) \quad \|\lambda x\| = |\lambda|\|x\| \quad for\ every\ x \in X\ and\ \lambda \in R$$

$$(\boldsymbol{S1}) \quad \|x\| \geq 0, \qquad and\ \|x\| = 0\ \ if\ and\ only\ if\ x = 0$$

$$(\boldsymbol{S3}) \quad \|x+y\| \leq \|x\| + \|y\| \quad (triangle\ inequality)\ \forall\, x,y \in X$$

Remark: In case the function $\|.\|$ satisfies only the properties (S_2) and (S_3), Then it is called a *seminorm* on

Definition 2.2.3 (Normed Space)A vector space with a norm is called $normedspace$.

Also we can say that a $normedspace$ is a pair$(\boldsymbol{X}, \|.\|)$, where X is a vector space and $\|.\|$ is a norm defined on X.

2.3 Properties of Normed Spaces

2.3.1 Sequences

*Sequence***.** A *Sequence* in a normed space X is an ordered set in X whose members can be labelled with positive integers. We write $\{u_1, u_2, \dots\}$ or $\{u_k\}_{k=1}^{\infty}$.

*Convergence of sequences***.** A sequence $\{u_n\}$ in a subset Y of a normed space X is*convergent* if there is a member $u \in Y$ for which, given any $\epsilon > 0$, a number N can be found such that

$$\|u_n - u\| < \epsilon \qquad for\ all\ \ n > N$$

If this is the case, we write $u_n \longrightarrow u$ (which is read 'u_n $converges\ to\ u$') and u is called the limit of the sequence. So, in other way we can express it by

$$\lim_{n\to\infty} \|u_n - u\| = 0 \quad or \quad \lim_{n\to\infty} u_n = u$$

Which is read '***the limit as n tends to*** ∞ ***of*** $\boldsymbol{u_n}$***, is u***'.

The convergence in a normed space has the basic properties of the convergence in R.

(*i*)A *convergent sequence has a unique limit*

$$\text{(ii)}\ If\ \ x_n \longrightarrow x \ \ \lambda_n \longrightarrow \lambda \ \ (\lambda, \lambda_n\ are\ scalars), \qquad then\ \lambda_n x_n \longrightarrow \lambda$$

$$(\boldsymbol{iii})\ \ If\ \ x_n \longrightarrow x\ \ and\ \ y_n \longrightarrow y, \qquad then\ x_n + y_n \longrightarrow x + y$$

Remark. A norm in a vector space ***E*** induces a convergence in ***E***. In other words, if we have a normed space ***E***, then we automatically have a convergence defined in ***E***.

*Uniform convergence:*Let $f,\ f_1,\ f_2\ , \dots, \in\ \boldsymbol{\ell(A)}$. We say that the sequence $\{f_n\}$*convergence uniformly* to f if for every $\epsilon > 0$ there exist a constant M such that for all $x \in A$ and for all indices $n \geq M$ we have

$$|f(x) - f_n(x)| < \epsilon$$

Where $\ell(A)$ is the space of all continuous functions defined on a closed bounded set $A \subset R^n$.
Pointwise convergence: Let. We say that the sequence $\{f_n\}$ is ***Pointwise convergence*** to f if

$$f_n(t) \longrightarrow f(t) \quad for\ every\ t \in [a,b]$$

Where $\ell([0,1])$ is the space of all continuous functionsdefined on the interval [0,1].

2.3.2 Completeness

Definition 2.3.2.1. A sequence $\{u_n\}$ in a subset Y of a normed space X is called a ***Cauchy sequence*** if

$$\lim_{m,n\to\infty} \|u_m - u_n\| = 0$$

or, in other words, more formally, if for any given $\epsilon > 0$ there exist a number N such that

$$\|u_m - u_n\| < \epsilon \quad whenever\ m,n > N$$

Definition 2.3.2.2. Let X be a normed space and let $\{x_n\}_n \subset X$ be a sequence. Then (x_n) is said to be *bounded* if there is a constant $c > 0$ such that,

$$\|x_n\| \le c \quad \forall\, n \in N$$

Definition 2.3.2.3. A normed space X is called *complete* if every Cauchy sequence in X converges to an element of X.

Definition 2.3.2.4. A *Branch Space* is a normed space which is complete in the metric defined by its norm; this meant that every Cauchy sequence is required to converge.
Proposition 2.3.2.1. Let X, Y be normed spaces and let $Z \subset X$ be a vector subspace. Then, the space $X \times Y$ becomes with the norm

$$\|(x,y)\| := (\|x\|_X^2 + \|y\|_Y^2)^{\frac{1}{2}}$$

is a normed space.

Proof. Note that coordinate wise addition/multiplication.

$$\alpha(x',y') + \beta(x,y) := (\alpha x' + \beta x, \alpha y' + \beta y)$$

Makes $X \times Y$ a vector space (S_1) and (S_2) ofa norm are obvious.
For (S_3)We consider $(x,y)$and$(x',y') \in X \times Y$.Then

$$\|(x,y) + (x',y')\| = (\|x + x'\|_X^2 + \|y + y'\|_Y^2)^{\frac{1}{2}}$$

$$\le [(\|x\|_X + \|x'\|_X)^2 + (\|y\|_Y + \|y'\|_Y)^2]^{\frac{1}{2}}$$

$$= [\|x\|_X^2 + \|x'\|_X^2] + 2\|x\|_X\|x'\|_X + \|y\|_Y^2 + \|y'\|_Y^2 + 2\|y\|_Y\|y'\|_Y{}^{\frac{1}{2}}$$

$$\le \left[\|x\|_X^2 + \|y\|_Y^2 + 2\sqrt{\|x\|_X^2 + \|y\|_Y^2}\ .\ \sqrt{\|x'\|_X^2 + \|y'\|_Y^2}\ +\ \|x'\|_X^2 + \|y'\|_Y^2\right]^{\frac{1}{2}}$$

$$= \left[(\|x\|_X^2 + \|y\|_Y^2)^{\frac{1}{2}} + (\|x'\|_X^2) + \|y'\|_Y^2)^{\frac{1}{2}}\right]^{2\cdot\frac{1}{2}}$$

$$= \|(x,y)\| + \|(x',y')\|$$

Definition 2.3.2.5. Let X be a normed space and let $(x_n)_n \subset X$ be a sequence. Then,

$$(i) \quad x_n\ converges\ to\ x \leftrightarrow lim_{n\to\infty}\|x_n - x\| = 0$$

$$(ii)\quad \sum_{n=1}^{\infty} x_n \text{ converges if the sequence of partial sums } s_N$$

$$= \sum_{n=1}^{N} x_n \text{ converges}$$

Theorem 2.3.2.1.A $closedvectorsubspaceofaBanach\ spaceisaBanach\ spaceitself.$

Proof.Let $(E, \|.\|)$ be a Banach space and let F be a closed vector subspace of E. If $\{x_n\}$ is a Cauchy sequence in, then it is a Cauchy sequence in E and therefore there exists $x \in E$ such that $x_n \longrightarrow x$. Since F is a closed subset of E, we have $x \in F$. Thus every Cauchy sequence in F converges to an element of F.

Proposition 2.3.2.2.A Cauchy sequence is bounded.

Proof.Let $(x_m)_m$be a Cauchy,

$$\exists N = N_1 \quad : \quad \forall n, m \geq N \quad : \quad \|x_n - x_m\| \leq 1. \ Therefore,$$

$$\|x_n\| \leq \|x_n - x_N\| + \|x_N\| \leq 1 + \|x_N\| \quad \forall n \geq N$$

So, we can find

$$\|x_m\| \leq max\,\{\|x_1\|, \|x_2\|, \ldots\ldots, \|x_N\|, \|x_N\| + 1\} < \infty \quad \forall m.$$

Let Let X be a metric space, $x \in X$ and $r > 0$. We call**Definition 2.3.2.6.**

$B_r(x) = \{y \in X: d(x,y) < r\}\ open\ ball$

$K_r(x) = \{y \in X: d(x,y) \leq r\}\ closed\ ball$

with centre x, radius $r > 0$.

Proposition 2.3.2.3. Let X, Y be complete metric spaces (respectively, Banach spaces) and $Z \subset X$ be a closed subset (respectively, a closed vector subspace).

$(i)\ \ X \times Y$ *is complete metric space (Banach space)*

$(ii)\ \ Z$ *is a complete metric space (Banach space)*

Theorem 2.3.2.2 (BaireTheorem). Let X be a complete metric space and $(D_j)_{j\in\mathbb{N}}$ be countably many dense open subsets. Then $D := \bigcap_{j\in\mathbb{N}} D_j$is dense.

2.4 Linear Mappings

Definition 2.4.1 (Linear maps). Let E and F be K- vector spaces. A *linear mapA* of E into F is a map $A : E \longrightarrow F$ such thatLet X, Y be normed spaces and let $A : X \longrightarrow Y$ be a linear map. Then the following are equivalent:

$$(\lambda x + \mu y) = \lambda A(x) + \mu A(y) \quad for\ all\ x, y \in E\ and\ \lambda, \mu \in K$$

A linear map $A : E \longrightarrow K$ is called a *linear functional*or also, a *linear form E*.
The set of all linear forms on $E$is called the algebraic dual E^* of E.
E^* becomes a K- vector space with the following definitions of addition and scalar multiplication:

$$y + z : x \longmapsto y(x) + z(x),\ \ \lambda y : x \longmapsto \lambda y(x)\,; \quad y, z \in E^*, \qquad \lambda \in K, x \in E$$

The zero map $0 : X \longrightarrow Y$ mapping every element $x \in X$ to $0 \in Y$.

مجلة التربوي
Journal of Educational
ISSN: 2011- 421X
Arcif Q3
معامل التأثير العربي 1.5
العدد 19

Proposition 2.4.1.

(i)A linear map $A : X \longrightarrow Y$ is called *injective*if for all $x_1, x_2 \in X$, the condition $Ax_1 = Ax_2$ implies that $x_1 = x_2$. In other words, different vectors in X are mapped to different vectors in Y.

(ii) A linear map $A : X \longrightarrow Y$ is called *surjective*if the range of $A = Y$.

(iii) A linear map is called *bijective* if $A$is *injective* and *surjective.*

Definition 2.4.2 (continuous mappings). Let E_1 and E_2 be two normed spaces, and let L be a mapping from E_1 into E_2. If for any sequence $\{x_n\}$ of elements of E_1 converges to $x_0 \in E_1$, the sequence $\{L(x_n)\}$ converges to $L(x_0)$, the mapping L is called *continuous*at x_0, i.e., L is continuous at x_0 if

$$\|x_n - x_0\| \longrightarrow 0 \qquad implies \qquad \|L(x_n) - L(x_0)\| \longrightarrow 0$$

$$\text{or } x_n \longrightarrow x_0 \;\; in \;\; E_1 \quad implies \quad Lx_n \longrightarrow Lx_0 \;\; in \;\; E_2$$

If L is continuous at every $x \in E_1$, then we simply say that L is continuous.
We can also define the continuous mapping on a normed space by saying that:
Let E be a normed space. A map $A : E \longrightarrow E$ is said to be *continuous*if for $\epsilon > 0$, there exist $\delta > 0$ such that

$$\|x - y\| < \delta \qquad implies \qquad \|A(x) - A(y)\| < \epsilon \qquad for\ all\ x, y \in E$$

Theorem 2.4.1. *A linear mapping $L : E \longrightarrow F$ is a continuous if and only if it is continuous at a point.*

Proof. Assume L is continuous at $x_0 \in E$. Let x be arbitrary element of E and let $\{x_n\}$ be a sequence convergent to x. Then the sequence $\{x_n - x + x_0\}$ converges to x_0 and thus we have

$$\|L(x_n) - L(x)\| = \|L(x_n - x + x_0) - L(x_0)\| \longrightarrow 0$$

Definition 2.4.3 (Bounded linear mapping).
A linear mapping $L : E \longrightarrow F$ is said to be bounded if there is a number C such that

$$\|L(x)\| \leq C\|x\| \qquad for\ all\ x \in E$$

Proposition 2.4.2. Let X, Y be normed spaces and let $A : X \longrightarrow Y$ be a linear map. Then the following are equivalent:

a) A is continuous.

b) A is bounded. That is,

$$c := sup_{u \in X_{u \neq o}} \frac{\|Au\|}{\|u\|} < \infty$$

$$Where \;\; c = sup_{u \neq o} \frac{\|A_u\|}{\|u\|},$$

$$u \in X \;\; is\ the\ norm\ of\ the\ linear\ map \;\; A$$

***Proof*. $b) \Rightarrow a)$** If $c < \infty$, we get

$$\|Au_n - Au\| = \|A(u - u_n)\| \leq c \,.\, \|u - u_n\|$$

$$thus, \quad u_n \longrightarrow u \;\; implies \;\; Au_n \longrightarrow Au.$$

مجـلة الـتربـوي
Journal of Educational
ISSN: 2011- 421X
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معامل التأثير العربي 1.5
العدد 19

$\boldsymbol{a}) \Rightarrow \boldsymbol{b})$ Assume A were unbounded : $c = \infty$. Then

$$\forall\, n \in \mathbb{N} \quad \exists\, u_n \in X \quad : \quad \|Au_n\| > n\,.\|u_n\|.$$

$$Set\ \ v_{n:} := \frac{u_n}{n\|u_n\|}\ \ and\ observe\ that$$

$$\|v_n\| = \frac{\|u_n\|}{n\|u_n\|} = \frac{1}{n} \longrightarrow 0.$$

$$Thus\ \ v_n \longrightarrow 0\ and\ \|Av_n\| = \left\|A\frac{u_n}{n\|u_n\|}\right\| = \frac{\|Au_n\|}{n\|u_n\|}\ \ i.e.\ \ \|Av_n\| > \frac{n\|u_n\|}{n\|u_n\|} = 1$$

$$i.e.\ \ Av_n \nrightarrow A0 = 0.$$

Definition 2.4.4. A mapping A from a subset E of a normed space F into $F(A : F \longrightarrow F)$ is called a *contracting mapping*(or simply a*contraction*) if there exist a positive number *c*< 1 such that

$$\|Au - Aw\| \le c\|u - w\| \qquad for\ every\ \ u, w \in E$$

Theorem 2.4.2. (*Banach Fixed Point / Contraction Mapping Principle*).

Let X be any complete metric space (Banach space) let $F \subset X$ be a closed subset and $A : F \rightarrow F$ be a contraction, i.e., a map satisfying

$$\|Au - Aw\| \le c\|u - w\| \quad \forall u, w \in F$$

Where *c* is a constant belonging to the unit open interval (0,1). Then, there is a unique fixed point $f \in F$, i.e., a point with $Af = f$.

Proof the uniqueness.Assume that f, g are two fixed points with $f \neq g$. Then

$$As\ f = Af\ \ and\ g = Ag\ from\ the\ uniqueness\ \Longrightarrow$$

$$0 < \|f - g\| = \|Af - Ag\|$$

$$\le c\|f - g\| \quad < \|f - g\| \quad contradiction\,!\,since\ \|f - g\|\ can\ not\ be\ less\ than\ \|f - g\|$$

$$\Longrightarrow \quad \|f - g\| = 0 \quad \Longrightarrow f = g$$

So, there is just one unique fixed point.

Proof the existence.*Write* $A^n u$ *for* $A \circ A \circ A \circ \ldots \ldots \circ A$ *and* $A^o u = u$.

We want to prove that $(A^n u)_n$ *is Cauchy*

One we get this; we conclude that$f := \lim\limits_{n} A^n u$*exists (by completeness)and it is a fixed point. Indeed.*

$$Af = A\left(\lim_n A^n u\right) = \lim_n A(A^n u) = \lim_n A^{n+1} u = f$$

Now, to prove the sequence $(A^n u)_n$ is really Cauchy: take $n, k \in \mathbb{N}$. Then

$$\|A^n u - A^{n+k}u\| = \|A^n u - A^n A^k u\|$$

$$\le c^n \|u - A^k u\|$$

$$\le c^n(\|u - Au\| + \|Au - A^2 u\| + \cdots \ldots \ldots + \|A^{k-1}u - A^k u\|)\ \ triangle\ inequality$$

$$\le c^n(\|u - Au\| + c\|u - Au\| + c^2\|u - Au\| + \cdots \ldots + c^{k-1}\|u - Au\|)$$

$$= c^n(1 + c + c^2 + \cdots + c^{k-1})\ \|u - Au\|$$

$$\leq c^n(1 + c + c^2 + \cdots + c^{k-1} + \cdots)\,\|u - Au\|$$

$$= \frac{c^n}{1 - c}\|u - Au\|$$

Where we used the triangle inequality in the third line, the contraction property in the line 4, and the geometric series in the last line.

Proposition 2.4.2. The function

$$|||A||| := \quad sup_{u\in X_{u\neq 0}} \frac{\|Au\|}{\|u\|} = \quad sup_{w\in X_{\|w\|=1}} \|Aw\|$$

is a norm on $\mathcal{L}(X, Y)$.

where $\mathcal{L}(X, Y)$ is the space of all continuous linear maps from X to Y:

$$\mathcal{L}(X, Y) = \{A : X \longrightarrow Y : \ A\ linear, continuous\}.$$

Theorem 2.4.3(Open Mapping Theorem).

Let X, Y be Branch spaces, and let $A : X \longrightarrow Y$ be a bounded, surjective linear map. Then A maps open sets in X to open sets in Y.

Theorem 2.4.4(Inverse Mapping Theorem).

If A is a bounded linear bijection from a Branch space X to Branch space Y, then the inverse of A is continuous (and hence bounded).

Theorem 2.4.5(Closed Graph Theorem).

Let $A : X \longrightarrow Y$ be a linear map between Branch spaces X and Y. Then the graph A is closed if and only if A is bounded.

Given a map $A : X \longrightarrow Y$, then the graph of A is the set.

$$G(A) = \{\, (x, y) \in X \times Y \ : \ y = Ax \,\}.$$

Note that if X and Y are normed linear spaces, and A is linear, then $G(A)$ is a linear subspace of $X \times Y$, with norm $\|x, y\|_{G(A)} = \|x\|_X + \|y\|_Y$.

Proof. *Suppose first that A is bounded, and let $\{(x_n, Ax_n)\}$ be a sequence in G(A) with limit $(x, y) \in X \times Y$. Since A is bounded, then*

$$\|A(x_n - x)\|_y \leq \|A\|\|x_n - x\|_X \longrightarrow 0, \qquad and\ so\ \ Ax_n \longrightarrow Ax.$$

We also have

$$\|(x_n, Ax_n) - (x, y)\|_{X\times Y} \longrightarrow 0$$

$$\Leftrightarrow \quad \|x_n - x\|_X + \|Ax_n - y\|_Y \longrightarrow 0$$

$$\Rightarrow \quad Ax_n \longrightarrow y.$$

Uniqueness of limits then implies that $y = Ax$, and so $(x, y) \in G(A)$. Therefore $G(A)$ is closed.

Now suppose that $G(A)$ is closed. Then it is closed linear subspace of $X \times Y$, and in particular it is a Banach space.

We have two projection maps

$$P_1 : G(A) \longrightarrow X \qquad P_2 : G(A) \longrightarrow Y$$
$$(x.Ax) \longmapsto x \qquad (x, Ax) \longmapsto Ax$$

The following estimates show that P_1 and P_2 are bounded.

$$\|P_1(x, Ax)\|_Y = \|x\|_X \leq \|x\|_X + \|Ax\|_Y = \|(x, Ax)\|_{G(A)}$$

$$\|P_2(x, Ax)\|_Y = \|Ax\|_Y \leq \|x\|_X + \|Ax\|_Y = \|(x, Ax)\|_{G(A)}$$

Since P_1 is a bounded linear projection, then P_1^{-1} is bounded by Theorem 6 (Inverse Mapping Theorem) (note that this uses the Open Mapping Theorem, and hence the Baire Category Theorem). Therefore $A = P_2\ o\ P_1^{-1}$ is bounded, which is complete the proof.

3. HILBERT SPACES

3.1 Some definitions and overview about Hilbert Spaces

In order to get some results that could provide help for the next sections, we need to define some spaces such as Inner Product Space and Hilbert Space and discuss some relations between Inner Product and Normed Space with some properties of Hilbert space. This section will be devoted for defining the required spaces.

Definition 3.1.1. Let H be a vector space over the field $\mathbb{C}\ (K\ or\ \mathbb{R})$. A function $(x, y) \longrightarrow < x, y >$ from $H \times H$ into $\mathbb{R}\ resp\ \mathbb{C}$is called an $inner\ product$ on H if for all $x, y, z \in H$ and for all $\in \mathbb{R}\ or\ \mathbb{C}$, we have

$$(\boldsymbol{a}) \quad < x, x > \geq 0, and \ < x, x > = 0$$
$$if\ and\ only\ if\ x = 0$$
$$(\boldsymbol{b}) \quad < x + y, z > = < x, z > + < y, z >$$
$$(\boldsymbol{c}) \quad < \lambda x, y > = \lambda < x, y >$$
$$(\boldsymbol{d}) \quad < x, y > = \overline{< y, x >}$$

An*inner product*is also called a *scalar product*.

Definition 3.1.2. A vector space equipped with a given inner product $\|x\| = \sqrt{< x, x >}, \ \ \forall\ x \in H$ is called an inner product space (also called a*pre-Hilbert space*).

The pair $(H, < .,. >)$ is called the *inner product space.*

Proposition 3.1.1. From the definition of inner product space, we obtained some results as following,

$$(i) \quad < x, \lambda y > = \bar{\lambda} < x, y >$$
$$< x, \lambda y > = \overline{< \lambda y, x >} = \overline{\lambda < y, x >}$$
$$= \bar{\lambda}\, \overline{< y, x >} = \bar{\lambda} < x, y >$$

$$(ii) \quad < \lambda_1 x + \lambda_2 y, z > =$$

$$\lambda_1 < x, z > + \lambda_2 < y, z >.$$

$$< \lambda_1 x + \lambda_2 y, z > = \overline{< z, \lambda_1 x + \lambda_2 y >}$$

$$= \overline{\overline{\lambda_1}} \overline{< z, x >} + \overline{\overline{\lambda_2}} \overline{< z, y >}$$

$$= \lambda_1 < x, z > + \lambda_2 < y, z >$$

Proposition 3.1.2. Let X be a pre- Hilbert space, $\mathbb{R} = K$, and $x, y \in X$. Then

$$(i) \quad \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

$$(Parallelogram\ law)$$

$$(ii) \quad < x, y > = \frac{1}{4}\|x + y\|^2 - \frac{1}{4}\|x - y\|^2$$

$$(Polarisation\ identity)$$

$$(iii) \; < x, y > = 0 \implies \|x + y\|^2 = \|x\|^2 + \|y\|^2$$

$(Pythagoras)$

Theorem 3.1.1. The inner product $(x, y) \longrightarrow < x, y >$ from the product normed space$H \times H$ into $\mathbb{R}\ or\ \mathbb{C}$ is continuous.

Proof. Suppose $x_n \longrightarrow a$ and $y_n \longrightarrow b$ in H. Then we have

$$| < x_n, y_n > - < a, b > | =$$

$$| < x_n, y_n > - < x_n, b > + < x_n, b > - < a, b > |$$

$$= | < x_n, y_n - b > - < x_n - a, b > |$$

$$\leq | < x_n, y_n - b > | - | < x_n - a, b > |$$

$$\leq \|x_n\| \|y_n - b\| + \|b\| \|x_n - a\|$$

When $x_n \longrightarrow a$ then $\|x_n - a\| \longrightarrow 0$ as $n \longrightarrow \infty$

since $y_n \longrightarrow b$ Moreover, $\|y_n - b\| \longrightarrow 0$ as $n \longrightarrow \infty$

Now, as $\|x_n\| \|y_n - b\| + \|b\| \|x_n - a\| \longrightarrow 0$

$$\implies |< x_n, y_n > - < a, b >| \longrightarrow 0$$

And then, $< x_n, y_n > \longrightarrow < a, b >$

Definition 3.1.3. A *Hilbert Space* is a complete $\mathbb{R}$ *or* $\mathbb{C}$ - vector space equipped with a scalar (or inner) product.

From the definition we notice that a *pre-Hilbert Space*which is complete under the norm $\|x\| = \sqrt{< x,x >}$ is called a *Hilbert Space*.

3.2 Relation between norms and scalar product

All *inner product spaces*are also *normed vector spaces*. Given a scalar product $\|x\| := \sqrt{< x,x >} \quad \forall\, x \in H$ is a norm.
Conversely, a scalar product comes from a norm only if the norm satisfies the parallelogram law.
Theorem 3.2.1. Every inner product space H is a normed space under the norm

Proof.

(S_1)does hold, since $\sqrt{< x,x >} = 0$ when $\|x\| = 0$. Also, the other way around is correct.
(S_2) does also hold, since

$$= \lambda\bar{\lambda} < x,x > = |\lambda|^2 \|x\|^2$$

$$\Longrightarrow \quad \|\lambda x\| = |\lambda|\,\|x\| \qquad \forall\, x \in H, \lambda \in \mathbb{R} \text{ or } K$$

(S_3) In order to show the triangle inequality holds we consider

$$\|x+y\|^2 = < x+y, x+y > =$$
$$< x,x > + < x,y > + < y,x > + < y,y >$$
$$= < x,x > + 2Re < x,y > + < y,y >$$
$$\leq < x,x > + 2|< x,y >| + < y,y >$$
$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2 \Longrightarrow$$
$$\|x+y\| \leq \|x\| + \|y\|. \quad \forall\, x,y \in H$$

Then the triangle inequality holds and complete the proof.

3.3 Relation between Hilbert space and Banach space

A *Hilbert space* is a *Branch space* with norm given by an inner (scalar) product

$$\|x\| = \sqrt{< x,x >} \qquad (*)$$

For example, $(L^2(\mu), \|.\|_2)$ where
$\|x\|_2 = (\int \|x\|^2\, d\mu)^{\frac{1}{2}}$.
So, we showed that Every Hilbert space is a Banach space.
But the question that should be asking now, Is it the converse true also?
In other words, is Every Banach space also Hilbert space?
The answer is that it is a necessary and sufficient for a Banach space to be Hilbert is for the norm stated above $(*)$ should satisfy the parallelogram law.

3.4. Elementary Properties of Hilbert space

There are many inequalities and identities in Hilbert spaces those can provide some useful information to the operator theory in Hilbert spaces such as the Parallelogram identity, Polarization identity, Cauchy Schwarz inequality, The Pythagorean identity, etc. The parallelogram law plays a fundamental role in higher mathematics. As we just discussed in previous section the relation between *Hilbert space* and Banach, and we said that the essential condition for a Banach space to be Hilbert is achieving the parallelogram law.

3.4.1 Parallelogram Law

The definition of Parallelogram law can be defined as follows:
For any two elements x and y of an inner product space H (*Pre-Hilbert space*) we have

$$\|x+y\|^2+\|x-y\|^2=2(\|x\|^2+\|y\|^2)$$

Proof. $\|x+y\|^2+\|x-y\|^2 =$

$$<x+y,x+y> \; + \; <x-y,x-y>$$

$$= \; <x,x> + <x,y> + <y,x> + <y,y>$$

$$+ <x,x> - <x,y> - <y,x> + <y,y>$$

$$= \; 2<x,x> + \; 2<y,y>$$

$$As \; <x,x>= \|x\|^2 and \; <y,y>= \|y\|^2 \; thus,$$

$$\|x+y\|^2+ \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

$$= \; 2(\|x\|^2+\|y\|^2)$$

3.4.2 Polarization Identity

Let $x,y \in V$ be an *innerproduct space*. If V is a vector space over $\mathbb{R}$ then

$$4<x,y>= \|x+y\|^2 - \|x-y\|^2$$

If V is a vector space over $\mathbb{C}$ then

$$4<x,y>= \|x+y\|^2 - \|x-y\|^2$$
$$-i(\|x+iy\|^2-\|x-iy\|^2)$$

Proof. For a real vector space,

$$\|x+y\|^2-\|x-y\|^2 =$$

$$<x+y,x+y> - <x-y,x-y>$$

$$= <x,x>+<x,y>+<y,x>+<y,y>$$

$$- (<x,x>-<x,y>-<y,x>+<y,y>)$$

$$=2<x,y>+2<y,x>$$

$$=2<x,y>+2<x,y>$$

$$=4<x,y>.$$

For a complex vector space, note that by the same calculation as above.

$$\|x+y\|^2-\|x-y\|^2=2<x,y>+2<y,x>$$

$$\text{And, } \|x+iy\|^2 - \|x-iy\|^2 = 2 < x, iy > +2 < iy, x >$$

$$LHS = \; 2 < x, y > +2 < y, x > +$$

$$2i < x, iy > +2i < iy, x >$$

$$LHS = \; 2 < x, y > +2 < y, x > +$$

$$2i < x, iy > +2i < iy, x >$$

$$= 2 < x, y > +2 < y, x > +$$

$$2(-i)^2 \; < x, y > +2i^2 < y, x >$$

$$2 < x, y > +2 < y, x > +$$

$$2 < x, y > -2 < y, x >$$

$$= \; 4 < x, y >$$

3.4.3 Pythagorean Identity

Let $x, y \in H$ *, where* H *is a pre-Hilbert space, then*

$$< x, y > = 0 \Longrightarrow \; \|x+y\|^2 = \|x\|^2 + \|y\|^2$$

Proof. $\|x+y\|^2 = < x+y, x+y >$

$$= < x, x > + < x, y > + < y, x > + < y, y >$$

$$As \; < x, y > = 0 \;\; so \; is \; < y, x >. \; Then$$

$$\|x+y\|^2 =< x, x > +< y, y >$$

$$= \; \|x\|^2 + \|y\|^2$$

3.4.4 Cauchy-Schwarz Inequality

Let H *be a pre-Hilbert space. Then for all* $x, y \in H$*, we have*

$$Assume \; y \neq 0. \; Then \; < y, y > \neq 0.$$

$$Let \; \lambda = \frac{< x, y >}{< y, y >}, \forall \; \lambda \in \mathbb{C}.$$

Then we have

$$0 \leq \; \|x - \lambda y\|^2 = < x - \lambda y, x - \lambda y >$$

$$= < x, x > -\bar{\lambda} < x, y > -$$

$$\lambda < y,x > +\lambda\bar{\lambda} < y,y >$$

$$= < x,x > - \frac{< x,y >< y,x >}{< y,y >}.$$

Therefore, we have

$$|< x,y >|^2 = < x,y >< y,x >$$

$$\Longrightarrow 0 \leq < x,x > - \frac{|< x,y >|^2}{< y,y >}$$

$$\frac{\|< x,y >\|^2}{< y,y >} \leq < x,x >$$

$$\Longrightarrow |< x,y >|^2 \leq < x,x >< y,y >$$

$$= \|x\|^2\|y\|^2$$

By taking the square root for both sides, we get

$$|< x,y >| \leq \|x\|\|y\|$$

3.4.5 Minkowski inequality

Let H be a *pre-Hilbert space*and $,y\in H$. Then we have

$$\|x + y\| \leq \|x\| + \|y\|$$

Proof. $\|x + y\|^2 = < x + y, x + y >$

$$= < x,x > +< x,y > +< y,x > +< y,y > (1)$$

$$= \|x\|^2 +< x,y > +< y,x > +\|y\|^2$$

Now, from the Cauchy-Schwarz inequality just stated above.We have

$$|< x,y >| \leq \|x\|\|y\|$$

Then we can say that

$$< x,y > \leq \|x\|\|y\|$$

Now we apply these results in our main equation (1). We get

$$\|x + y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

$$\Longrightarrow \|x + y\| \leq \|x\| + \|y\|$$

3.4.6 Appolonius identity

Let x,y,z be three elements in a *pre-Hilbert space* called H.Then we have

$$\|z-x\|^2+\|z-y\|^2=$$

$$\frac{1}{2}\|x-y\|^2+\frac{1}{2}\|2z-(x+y)\|^2$$

Proof. $\|z-x\|^2+\|z-y\|^2=$

$$<z-x,z-x>+<z-y,z-y>$$

$$=<z,z>-<z,x>-<x,z>+<x,x>$$

$$+<z,z>-<z,y>-<y,z>+<y,y>$$

$$=2\|z\|^2+\|x\|^2+\|y\|^2-<z,x>$$

$$-<x,z>-<z,y>-<y,z>\ \rightarrow\ (*)$$

$$\|x-y\|^2+\|2z-(x+y)\|^2=$$

$$<x-y,x-y>+<2z-(x+y),2z-(x+y)>$$

$$=<x,x>-<x,y>-<y,x>+<y,y>$$

$$+<2z,2z>-<2z,x+y>$$

$$-<x+y,2z>+<x+y,x+y>$$

$$=\|x\|^2+\|y\|^2+\|2z\|^2-2<z,x+y>$$

$$-2<x+y,z>-<x,y>-<y,x>$$

$$+<x+y,x+y>$$

$$=\|x\|^2+\|y\|^2+\|2z\|^2-2<z,x>$$

$$-2<z,y>-2<x,z>-2<y,z>$$

$$-<x,y>-<y,x>+<x,x>$$

$$+<x,y>+<y,x>+<y,y>$$

$$=\|x\|^2+\|y\|^2+\|2z\|^2-2<z,x>$$

$$-2<z,y>-2<x,z>$$

$$-2 < y,z > +\|x\|^2 + \|y\|^2$$

$$= 2\|x\|^2 + 2\|y\|^2 + 4\|z\|^2 - 2 < z,x >$$

$$-2 < z,y > -2 < x,z > -2 < y,z >$$

$$therefore, we\ apply\ the\ results\ now$$

$$\frac{1}{2}\|x-y\|^2 + \frac{1}{2}\|2z-(x+y)\|^2 =$$

$$\frac{1}{2}(\|x-y\|^2 + \|2z-(x+y)\|^2)$$

$$= \|x\|^2 + \|y\|^2 + 2\|z\|^2 - < z,x >$$

$$- < z,y > - < x,z > - < y,z > \quad \rightarrow (**)$$

$$Now\ from\ (*),(**)\ we\ conclude\ that,$$

$$\|z-x\|^2 + \|z-y\|^2 =$$

$$\frac{1}{2}\|x-y\|^2 + \frac{1}{2}\|2z-(x+y)\|^2$$

$so, the\ prove\ is\ done\ and\ the\ inequality\ holds.$

Definition 3.4.1. Let V be an inner product space (pre- Hilbert space).

(i) Afamily S of non-zero vectors in V is called an orthogonal sequence if $x \perp y$for any two distinct elements of $S \iff < x,y > = 0$.

(ii) A collection of vectors $(x_\alpha)_{\alpha \in A} \subseteq V$ is said to be orthogonal sequence if $< x_\alpha, x_\beta > = 0$ for all $\alpha \neq \beta$ and if $< x_\alpha, x_\beta > = 1$ for all $\alpha = \beta$.

Theorem 3.4.1. Let $x_1, x_2, \ldots\ldots, x_n$ are orthogonal vectors in an inner productspace, then

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 = \sum_{k=1}^{n}\|x_k\|^2. \ (*)$$

Proof. $x_1 \perp x_2$, then $\|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2$ by identity (3.4.4). Thus, the theorem is true for$n = 2$. Assume now that the $(*)$holds for$n-1$, that is

$$\left\|\sum_{k=1}^{n-1} x_k\right\|^2 = \sum_{k=1}^{n-1}\|x_k\|^2.$$

$$Set\ x = \sum_{k=1}^{n-1} x_k \quad and \quad y = x_n.$$

$$Clearly\ x \perp y.\ Thus$$

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 = \|x+y\|^2 = \|x\|^2 + \|y\|^2$$

$$= \sum_{k=1}^{n-1} \|x_k\|^2 + \|x_n\|^2 = \sum_{k=1}^{n} \|x_k\|^2.$$

This theorem is called Pythagorean Formula.

Theorem 3.4.2 (Orthonormalization Process). Let $\{x_n\}$ be a sequence, finite or infinite, of linearly independent vectors in an inner product space H. Then there is an orthonormal sequence $\{e_n\}$ in H such that for each k, both $\{x_1, x_2, \dots \dots, x_k\}$ and $\{e_1, e_2, \dots \dots . e_k\}$ generate the same vector subspace of H.

Proof.Let $G(z_1, z_2, \dots \dots z_k)$ denote the vector subspace spanned by the vectors $z_1, z_2, \dots \dots, z_k \in H$ and let $e_1 = \frac{x_1}{\|x_1\|}$ .Suppose $e_1, e_2, \dots \dots, e_k$ have been constructed by induction.

Let $a_{k+1} = x_{k+1} - \sum_{k=1}^{k} < x_{k+1}, e_j > e_j$. Suppose to the contrary that $a_{k+1} = 0$. Then $x_{k+1} \in G(e_1, e_2, \dots \dots, e_k) = G(x_1, x_2, \dots \dots, x_k)$ gives the linear dependence of $x_1, x_2, \dots \dots, x_k, x_{k+1}$ which is a contradiction.

Hence, $a_{k+1} \neq 0$. Define $e_{k+1} = \frac{a_{k+1}}{\|a_{k+1}\|}$. clearly, $\|e_{k+1}\| = 1$.

Also for $1 \le p \le k$, we have $< e_{k+1}, e_p > =$

$$< x_{k+1}, e_p > - \sum_{j=1}^{k} < x_{k+1}, e_j >< e_j, e_p >$$

$$< x_{k+1}, e_p > - \sum_{j=1}^{k} < x_{k+1}, e_j > \delta_{jp} =$$

$$< x_{k+1}, e_p > - < x_{k+1}, e_p > = 0$$

Thus, $e_1, e_2, \dots \dots, e_k, e_{k+1}$ are orthogonal.

Now the following calculation completes the proof.

$$G(e_1, e_2, \dots, e_k, e_{k+1}) = G[G(e_1, e_2, \dots., e_k), e_{k+1}]$$

$$= G\left[G(e_1, \dots, e_k), x_{k+1} - \sum_{k=1}^{k} < x_{k+1}, e_j > e_j\right]$$

$$= G[G(e_1, e_2, \dots, e_k), x_{k+1}]$$

$$= G[G(x_1, x_2, \dots, x_k), x_{k+1}]$$

$$= G(x_1, x_2, \dots, x_k, x_{k+1}).$$

Lemma 3.4.1. let $\{x_n\}_{n=1}^{N}$ be an orthonormal system in an inner product space V and let $\{a_n\}_{n=1}^{N}$ be a finite sequence of scalars. Then for all$x \in V$, we have

$$\left\|x - \sum_{n=1}^{N} a_n x_n\right\|^2 = \|x\|^2$$

$$- \sum_{n=1}^{N} |< x, x_n >|^2 + \sum_{n=1}^{N} |a_n - < x, x_n >|^2$$

Proposition 3.4.1 (Bessel's Equality and Inequality).Let $(e_i)_{i \in N} \subset X$ be an orthonormal sequence in the pre-Hilbert space X.Then, for every $x \in X$ we have

$$\left\|x - \sum_{i=1}^{n} < x, e_i > e_i\right\|^2 =$$

$$\|x\|^2 - \sum_{i=1}^{n} |< x, e_i >|^2 \quad (1)$$

$$\sum_{i=1}^{n} |< x, e_i >|^2 \leq \|x\|^2 \qquad (2)$$

Proof.In view of the Pythagorean identity (3.4.3), we have

$$\left\|\sum_{i=1}^{n} < x, e_i > e_i\right\|^2 =$$

$$\sum_{i=1}^{n} \|< x, e_i > e_i\|^2 = \sum_{i=1}^{n} |< x, e_i >|^2$$

Hence,

$$0 \leq \left\|x - \sum_{i=1}^{n} < x, e_i > e_i\right\|^2 =$$

$$< x - \sum_{j=1}^{n} < x, e_j > e_j , x - \sum_{i=1}^{n} < x, e_i > e_i >$$

$$= < x, x > - < x, \sum_{i=1}^{n} < x, e_i > e_i >$$

$$- < \sum_{j=1}^{n} < x, e_j > e_j , x >$$

$$+ < \sum_{j=1}^{n} < x, e_j > e_j , \sum_{i=1}^{n} < x, e_i > e_i >$$

$$= \|x\|^2 - \sum_{i=1}^{n} \overline{< x, e_i >} < x, e_i >$$

$$- \sum_{j=1}^{n} < x, e_j > \overline{< x, e_j >}$$

$$+ \sum_{j=1}^{n} \sum_{i=1}^{n} \overline{< x, e_j >} < x, e_i > < e_j, e_i >$$

Now, since is orthonormal. Then,

$$0 \leq \|x\|^2 - 2\sum_{i=1}^{n} |< x, e_i >|^2 + \sum_{i=1}^{n} |< x, e_i >|^2 \ (3)$$

$$= \|x\|^2 - \sum_{i=1}^{n} |< x, e_i >|^2$$

$\Longrightarrow$ equality (1) holds. Thus, from (3)

$$\sum_{i=1}^{n} |< x, e_i >|^2 \leq \|x\|^2 \ =$$

$$\sum_{i=1}^{\infty} |< x, e_i >|^2 \quad as\ n \longrightarrow \infty$$

Which is complete the prove and the inequality (2) holds.

Theorem 3.4.3 (Riesz-Fischer). Let $(x_n)_{n\in N}$ be an ON-sequence (Orthonormal) in the Hilbert space$X$. Then,

$$\sum_{n=1}^{\infty} c_n x_n \ converges \Leftrightarrow \sum_{n=1}^{\infty} |c_n|^2 < \infty.$$

If this holds then we have

$$\left\| \sum_{n=1}^{\infty} c_n x_n \right\|^2 = \sum_{n=1}^{\infty} |c_n|^2$$

First of all, we need to show that $\sum_{n=1}^{i} c_n x_n$is a Cauchy sequence.

Let's assume that $A_i = \sum_{n=1}^{i} c_n x_n$ for the partial sum.
Now,by Pythagoras we get $\forall\, j < i$ the following

$$\|A_i - A_j\|^2 = < A_i - A_j, A_i - A_j >$$
$$= < \sum_{n=j+1}^{i} c_n x_n , \sum_{k=j+1}^{i} c_k x_k >$$
$$= \sum_{n=j+1}^{i} \sum_{k=j+1}^{i} c_n \bar{c}_k < x_n, c_k >$$
$$= \left\| \sum_{n=j+1}^{i} c_n x_n \right\|^2$$
$$= \sum_{n=j+1}^{i} |c_n|^2.$$

This shows

$$\{A_i\} \ \ is\ Cauchy \Leftrightarrow \lim_{j,k\longrightarrow\infty, j<k} \sum_{j+1}^{k} |c_n|^2 = 0$$
$$\Leftrightarrow \sum_{1}^{\infty} |c_j|^2 < \infty.$$

But as X is complete,

$\Leftrightarrow \quad \{A_i\} \ \ converges.$

Now, by the continuity of the normas $\|.\|$we mentioned early in proposition … we get

$$\left\| \sum_{n=1}^{\infty} c_n x_n \right\|^2 = \left\| \lim_{i} A_i \right\|^2$$
$$= \lim_{i} \|A_i\|^2$$
$$= lim_i \sum_{n=1}^{i} |c_n|^2 = \sum_{n=1}^{\infty} |c_n|^2$$

Theorem 3.4.4. Let$\{x_n\}_{n=1}^{\infty}$be an orthonormal system in a Hilbertspace H. Then the following are equivalent.

$$(1) \ \ \{x_n\} \ \ is\ complete.$$
$$(2) \ \ for\ all\ x \in H,$$
$$x = \sum_{n=1}^{\infty} < x, x_n > x_n.$$
$$(3) \ \ \|x\|^2 = \sum_{n=1}^{\infty} |< x, x_n >|^2$$

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Journal of Educational
ISSN: 2011- 421X
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معامل التأثير العربي 1.5
العدد 19

$(Parseval\ identity)$
$(4)\quad for\ all\ x, y \in H, we\ have,$

$$< x, y > = \sum_{n=1}^{\infty} < x, x_n > \overline{< y, x_n >}$$

Proof. Let $x \in H$. By Bessel's Equality (1) proposition 3.4.1. For every $n \in N$, we have

$$\left\| x - \sum_{i=1}^{n} < x, e_i > e_i \right\|^2$$

$$= \|x\|^2 - \sum_{i=1}^{n} |< x, e_i >|^2 \quad (*)$$

Let $\{x_n\}_n = \{e_i\}_i$

$$\Longrightarrow \quad \left\| x - \sum_{n=1}^{k} < x, x_n > x_n \right\|^2$$

$$= \|x\|^2 - \sum_{n=1}^{k} |< x, x_n >|^2$$

If $\{x_n\}$ is complete,then the expression on the left in (*) converges to 0 as$k \longrightarrow \infty$.
Hence,

$$\lim_{k \longrightarrow \infty} \left[\|x\|^2 - \sum_{n=1}^{k} |< x, x_n >|^2 \right] = 0$$

Therefore (3) (Parseval identity) holds.

4. CONCLUSION:

In this paper we have introduced the Hilbert spaces along with their properties, operations, and applications. Specifically, we have focused on significant and crucial spaces for Hilbert spaces called Banach Spaces. We have also studiedthe normed spaces and their properties. Finally, we discussed what is meant by Hilbert spaces and what is the relation between Hilbert and Banach spaces. In Future, we are planning to extend this work by exploring the orthogonal complement, projection and Rieszrepresentation Theorem.

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