معامل التأثير العربي 1.5 العدد 19	مجـلـة الـتـربـوي Journal of Educational ISSN: 2011- 421X Arcif Q3	

مجلة التربوي

مجلة علمية محكمة تصدر عن كلية التربية

جامعة المرقب

العدد التاسع عشر

يوليو 2021م

هيئـــة تحرير

مجلة التربوي

- المجلة ترحب بما يرد عليها من أبحاث وعلى استعداد لنشرها بعد التحكيم .
- المجلة تحترم كل الاحترام آراء المحكمين وتعمل بمقتضاها .
- كافة الآراء والأفكار المنشورة تعبر عن آراء أصحابها ولا تتحمل المجلة تبعاتها .
- يتحمل الباحث مسؤولية الأمانة العلمية وهو المسؤول عما ينشر له .
- البحوث المقدمة للنشر لا ترد لأصحابها نشرت أو لم تنشر .

مجـلة الـتربـوي
Journal of Educational
ISSN: 2011- 421X
Arcif Q3

معامل التأثير العربي 1.5
العدد 19

ضوابط النشر:

يشترط في البحوث العلمية المقدمة للنشر أن يراعى فيها ما يأتي :

- أصول البحث العلمي وقواعده .
- ألا تكون المادة العلمية قد سبق نشرها أو كانت جزءا من رسالة علمية .
- يرفق بالبحث تزكية لغوية وفق أنموذج معد .
- تعدل البحوث المقبولة وتصحح وفق ما يراه المحكمون .
- التزام الباحث بالضوابط التي وضعتها المجلة من عدد الصفحات ، ونوع الخط ورقمه ، والفترات الزمنية الممنوحة للتعديل ، وما يستجد من ضوابط تضعها المجلة مستقبلا .

تنبيهات :

- للمجلة الحق في تعديل البحث أو طلب تعديله أو رفضه .
- يخضع البحث في النشر لأولويات المجلة وسياستها .
- البحوث المنشورة تعبر عن وجهة نظر أصحابها ، ولا تعبر عن وجهة نظر المجلة .

Information for authors

1- Authors of the articles being accepted are required to respect the regulations and the rules of the scientific research.
2- The research articles or manuscripts should be original and have not been published previously. Materials that are currently being considered by another journal or is a part of scientific dissertation are requested not to be submitted.
3- The research articles should be approved by a linguistic reviewer.
4- All research articles in the journal undergo rigorous peer review based on initial editor screening.
5- All authors are requested to follow the regulations of publication in the template paper prepared by the editorial board of the journal.

Attention
1- The editor reserves the right to make any necessary changes in the papers, or request the author to do so, or reject the paper submitted.
2- The research articles undergo to the policy of the editorial board regarding the priority of publication.
3- The published articles represent only the authors' viewpoints.

مجلة التربوي
Journal of Educational
ISSN: 2011- 421X
Arcif Q3

معامل التأثير العربي 1.5
العدد 19

On Coefficient Bounds for Certain Classes of Analytic Functions

S. M. Amsheri
Department Of Mathematics, Faculty of Science
Elmergib University, Libya
somia_amsheri@Yahoo.Com

N. A. Abouthfeerah
Department Of Mathematics, Faculty of Science
Al-Asmarya Islamic University, Libya
norianooh@Yahoo.Com

Abstract

In the present paper we obtain Fekete-Szegö inequalities and sharp bounds for certain classes of analytic and p-valent functions in the open unit disk involving certain fractional derivative operator.

Keywords: p-valent function , subordination , starlike function , convex function, fractional derivative operator, Fekete-Szegö inequality.

1. Introduction and Preliminaries

Let $A(p)$ denote the class of functions defined by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}\, z^{p+n}, \qquad (p \in \mathbb{N}) \tag{1.1}$$

which are analytic and p-valent in the open unit disk $\mathcal{U} = \{z: |z| < 1\}$.

Let $f(z)$ and $g(z)$ be functions analytic in $\mathcal{U}$, we say that the function $f(z)$ is subordinate to $g(z)$, if there exists a Schwarz function $w(z)$, analytic in $\mathcal{U}$, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathcal{U})$, and such that $f(z) = g\big(w(z)\big)$ for all $z \in \mathcal{U}$.

This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$. It is well known that, if the function $g(z)$ is univalent in $\mathcal{U}$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Let $\phi(z)$ be an analytic function with $\phi(0) = 1$, $\phi'(0) > 0$ and $\mathrm{Re}(\phi(z)) > 0$ $(z \in \mathcal{U})$, which maps the open unit disk $\mathcal{U}$ onto a region starlike with respect to 1 and is symmetric with respect to the real axis . Ali et al. [1] defined and studied the class $S^*_{b,p}(\phi)$ to be the class of functions $f(z) \in A(p)$ for which

$$1 + \frac{1}{b}\left\{\frac{1}{p}\,\frac{zf'(z)}{f(z)} - 1\right\} \prec \phi(z), \qquad (z \in \mathcal{U}, b \in \mathbb{C}\backslash\{0\})$$

and the class $C_{b,p}(\phi)$ of all functions for which

$$1-\frac{1}{b}+\frac{1}{bp}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\prec\phi(z),\qquad (z\in\mathcal{U}, b\in\mathbb{C}\backslash\{0\})$$

Note that $S_{1,1}^{*}(\phi)=S^{*}(\phi)$ and $C_{1,1}(\phi)=C(\phi)$, The classes were introduced and studied by Ma and Minda [4]. The familiar class $S^{*}(\alpha)$ of starlike functions of order α and the class $C(\alpha)$ of convex functions of order α, $0\le\alpha<1$ are the special cases of $S_{1,1}^{*}(\phi)$ and $C_{1,1}(\phi)$, respectively, when

$$\phi(z)=\frac{1+(1-2\alpha)z}{1-z}$$

Let ${}_2F_1(a,b;\ c;\ z)$ be the Gauss hypergeometric function defined for $z\in\mathcal{U}$ by, (see Srivastava and Karlsson [9]).

$${}_2F_1(a,b;\ c;\ z)=\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n\,n!}z^n$$

where $(\lambda)_n$ is the Pochhammer symbole defined, in terms of the Gamma function, by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}1, & n=0,\\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1), & n\in\mathbb{N}.\end{cases}$$

for $\lambda\neq 0,-1,-2,\ldots$.

We recall the following definitions of fractional derivative operators which were used by Owa [6], (see also [8]) as follows:

Definition 1.1. The fractional derivative operator of order λ is defined by

$$D_z^{\lambda}f(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z\frac{f(\xi)}{(z-\xi)^{\lambda}}d\xi\,,\qquad 0\le\lambda<1 \tag{1.2}$$

where $f(z)$ is analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi>0$.

Definition 1.2. Let $0\le\lambda<1$, and $\mu,\eta\in\mathbb{R}$. Then, in terms of the familiar Gauss's hypergeometric function ${}_2F_1$, the generalized fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}$ is

$$J_{0,z}^{\lambda,\mu,\eta}f(z)=\frac{d}{dz}\left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)}\int_0^z(z-\xi)^{-\lambda}f(\xi)\ {}_2F_1\left(\mu-\lambda,1-\eta;1-\lambda;1-\frac{\xi}{Z}\right)d\xi\right)\ \ (1.3)$$

where $f(z)$ is analytic function in a simply- connected region of the z-plane containing the origin, with the order $f(z) = O(|z|^{\varepsilon})$, $z \to 0$, where $\varepsilon > \max\{0, \mu - \eta\} - 1$ and the multiplicity of $(z - \xi)^{-\lambda}$ is removed requiring $\log(z - \xi)$ to be real when $z - \xi > 0$.

Definition 1.3 Under the hypotheses of Definition 1.2, the fractional derivative operator $J_{0,z}^{\lambda+m,\mu+m,\eta+m}$ of a function $f(z)$ defined by

$$J_{0,z}^{\lambda+m,\mu+m,\eta+m} f(z) = \frac{d^m}{dz^m} J_{0,z}^{\lambda,\mu,\eta} f(z) \tag{1.4}$$

Notice that

$$J_{0,z}^{\lambda,\lambda,\eta} f(z) = D_z^{\lambda} f(z), \qquad 0 \le \lambda < 1 \tag{1.5}$$

Amsheri and Zharkova [2], (see also [3], [10]) defined the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta,p} f(z)$ by

$$M_{0,z}^{\lambda,\mu,\eta,p} f(z) = \frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\mu)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)} z^{\mu} J_{0,z}^{\lambda,\mu,\eta} f(z)$$

$$= z^p + \sum_{n=1}^{\infty} \gamma_n(\lambda,\mu,\eta,p) a_{p+n} z^{p+n} \tag{1.6}$$

for $f(z) \in A(p)$ and $\lambda \ge 0$; $\mu < p+1$; $\eta > \max(\lambda, \mu) - p - 1$; $p \in \mathbb{N}$, where

$$\gamma_n(\lambda,\mu,\eta,p) = \frac{(p+1)_n (p+1-\mu+\eta)_n}{(p+1-\mu)_n (p+1-\lambda+\eta)_n}, \qquad (n \in \mathbb{N}) \tag{1.7}$$

Very recently, Zayed et al. [10] defined the operator $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}(z) : A(p) \to A(p)$, for $m \in \mathbb{N}_o = \mathbb{N} \cup \{0\}$ and $\delta \ge 0$ as follows:

$$N_{0,z}^{0,\lambda,\mu,\eta,\delta,p} f(z) = M_{0,z}^{\lambda,\mu,\eta,p} f(z)$$

$$N_{0,z}^{1,\lambda,\mu,\eta,\delta,p} f(z) = N_{0,z}^{\lambda,\mu,\eta,\delta,p} f(z)$$

$$= (1-\delta) M_{0,z}^{\lambda,\mu,\eta,p} f(z) + \delta \frac{z}{p} \left(M_{0,z}^{\lambda,\mu,\eta,p} f(z) \right)'$$

$$= z^p + \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p} \right) \gamma_n(\lambda,\mu,\eta,p) a_{p+n} z^{p+n}$$

and (in general),

$$N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z) = N_{0,z}^{\lambda,\mu,\eta,\delta,p} (N_{0,z}^{m-1,\lambda,\mu,\eta,\delta,p} f(z))$$

$$= z^p + \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p) a_{p+n} z^{p+n} \qquad (1.8)$$

We let $\gamma_n(\lambda,\mu,\eta,p) \equiv \gamma_n$.

Motivated essentially by aforementioned works, we introduce a more general class of p-valent analytic functions of complex order $S^m_{b,p,\lambda,\mu,\eta,\delta}(\phi)$ which we define in the following.

Definition 1.4 Let $\phi(z)$ be an univalent starlike function with respect to 1 which maps the open unit disk $\mathcal{U}$ onto a region in the right half-plane and symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(z) > 0$. A function $f(z) \in A(p)$ is in the class $S^m_{b,p,\lambda,\mu,\eta,\delta}(\phi)$ if

$$1 + \frac{1}{b}\left(\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p} f(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)} - 1\right) \prec \phi(z), \quad (z \in \mathcal{U}, b \in \mathbb{C}\backslash\{0\}) \qquad (1.9)$$

Also, we let $S^m_{1,p,\lambda,\mu,\eta,\delta}(\phi) = S^m_{p,\lambda,\mu,\eta,\delta}(\phi)$.

The above class $S^m_{b,p,\lambda,\mu,\eta,\delta}(\phi)$ is of special interest and it contains many well-known classes of analytic functions. In particular; for $m = \lambda = \mu = 0$, we have

$$S^0_{b,p,0,0,\eta,\delta}(\phi) = S^*_{b,p}(\phi)$$

where $S^*_{b,p}(\phi)$ is precisely the class which was studied by Ali et al. [1].

Furthermore, by specializing the parameters $b, p, \lambda, \mu, \delta$ and m we obtain the following subclasses which were studied by various other authors:

1- For $b = 1, p = 1$ and $\lambda = \mu = \delta = 0$, we get the class $S^m_{1,1,0,0,\eta,0}(\phi) = S^*(\phi)$ which studied by Ma and Minda [4].
2- For $p = 1$ and $\lambda = \mu = \delta = 0$, we get the class $S^m_{b,1,0,0,\eta,0}(\phi) = S^*_b(\phi)$ which studied by Ravichandran et al. [7].
3- For $b = 1$ and $\lambda = \mu = \delta = 0$, we get the class $S^m_{1,p,0,0,\eta,0}(\phi) = S^*_p(\phi)$ which studied by Ali et al. [1].
4- For $m = 0$, we get the class $S^0_{b,p,\lambda,\mu,\eta,\delta}(\phi) = S^*_{b,\lambda,\mu.\eta,\delta,p}(\phi)$ which studied by Amsheri and Zharkova [2].

The motivation of this paper is to give a generalization of the Fekete-Szegö inequalities obtained by Ali et al [1], Ma and Minda [4], Ravichandran et al. [7] and also by Amsheri and Zharkova [2].

Let Ω be the class of analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + \cdots$$

in the open unit disk $\mathcal{U}$ satisfying $|w(z)| < 1$.

In order to prove our results, we need the following results which shall be used in the sequel.

Lemma 1.5 [1]**.** If $w \in \Omega$, then

$$|w_2 - t w_1^2| \leq \begin{cases} -t & if \quad t \leq -1, \\ 1 & if \quad -1 \leq t \leq 1, \\ t & if \quad t \geq 1. \end{cases}$$

when $t < -1$ or $t > 1$, the equality holds if and only if $w(z) = z$ or one of its rotations. If $-1 < t < 1$, then equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if

$$w(z) = z\frac{\lambda + z}{1 + \lambda z}, \quad (0 \leq \lambda \leq 1)$$

or one of its rotations, while for $t = 1$, the equality holds if and only if

$$w(z) = -z\frac{\lambda + z}{1 + \lambda z}, \quad (0 \leq \lambda \leq 1)$$

or one of its rotations .

Although the above upper bound is sharp, it can be improved as follows when $-1 < t < 1$:

$$|w_2 - t w_1^2| + (t + 1)|w_1|^2 \leq 1, \qquad (-1 < t \leq 0)$$

and

$$|w_2 - t w_1^2| + (1 - t)|w_1|^2 \leq 1, \qquad (0 < t < 1).$$

Lemma 1.6 [5]**.** If $w \in \Omega$, then for any complex number t,

$$|w_2 - t w_1^2| \leq \max(1, |t|).$$

The result is sharp for the functions $w(z) = z$ or $w(z) = z^2$.

2. Coefficient Bounds

By making use of Lemmas 1.5-1.6, we prove the following:

Theorem 2.1. Let $0 \leq \theta \leq 1$; $\lambda \geq 0$; $\mu < p+1$; $\eta > \max(\lambda, \mu) - p - 1$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $\delta \geq 0$ and $p \in \mathbb{N}$. Further, let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$, where B_n□s are real with $B_1 > 0$, $B_2 \geq 0$, and

$$\sigma_1 = \frac{[(B_2 - B_1) + (p-\mu)B_1^2]\gamma_1^2 \left(\frac{p+\delta}{p}\right)^{2m}}{2\gamma_2 B_1^2 (p-\mu)\left(\frac{p+2\delta}{p}\right)^m}, \tag{2.1}$$

$$\sigma_2 = \frac{[(B_2 + B_1) + (p-\mu)B_1^2]\gamma_1^2 \left(\frac{p+\delta}{p}\right)^{2m}}{2\gamma_2 B_1^2 (p-\mu)\left(\frac{p+2\delta}{p}\right)^m}, \tag{2.2}$$

$$\sigma_3 = \frac{\left(\frac{p+\delta}{p}\right)^{2m}[B_2 + (p-\mu)B_1^2]}{2\gamma_2 B_1^2 (p-\mu)\left(\frac{p+2\delta}{p}\right)^m}. \tag{2.3}$$

If $f(z)$ given by (1.1) belongs to $S^m_{1,p,\lambda,\mu,\eta,\delta}(\phi)$, then

$$|a_{p+2} - \theta a_{p+1}^2|$$

$$\leq \begin{cases} \dfrac{(p-\mu)}{2\gamma_2\left(\frac{p+2\delta}{p}\right)^m}\left(B_2 - \dfrac{(p-\mu)\left(2\gamma_2\theta\left(\frac{p+2\delta}{p}\right)^m - \gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}\right)}{\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}}B_1^2\right), & \theta \leq \sigma_1, \\ \dfrac{(p-\mu)B_1}{2\gamma_2\left(\frac{p+2\delta}{p}\right)^m}, & \sigma_1 \leq \theta \leq \sigma_2, \\ -\dfrac{(p-\mu)}{2\gamma_2\left(\frac{p+2\delta}{p}\right)^m}\left(B_2 - \dfrac{(p-\mu)\left(2\gamma_2\theta\left(\frac{p+2\delta}{p}\right)^m - \gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}\right)}{\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}}B_1^2\right), & \theta \geq \sigma_2. \end{cases} \tag{2.4}$$

Further, if $\sigma_1 \leq \theta \leq \sigma_3$, then

$$\left|a_{p+2}-\theta a_{p+1}^2\right|+\frac{\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}}{2\gamma_2 B_1(p-\mu)\left(\frac{p+2\delta}{p}\right)^m}$$

$$\cdot\left(1-\frac{B_2}{B_1}+\frac{(p-\mu)\left(2\gamma_2\theta\left(\frac{p+2\delta}{p}\right)^m-\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}\right)}{\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}}B_1\right)\left|a_{p+1}\right|^2\le\frac{(p-\mu)B_1}{2\gamma_2\left(\frac{p+2\delta}{p}\right)^m}\qquad(2.5)$$

If $\sigma_3\le\theta\le\sigma_2$, then

$$\left|a_{p+2}-\theta a_{p+1}^2\right|+\frac{\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}}{2\gamma_2 B_1(p-\mu)\left(\frac{p+2\delta}{p}\right)^m}$$

$$\cdot\left(1+\frac{B_2}{B_1}-\frac{(p-\mu)\left(2\gamma_2\theta\left(\frac{p+2\delta}{p}\right)^m-\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}\right)}{\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}}B_1\right)\left|a_{p+1}\right|^2\le\frac{(p-\mu)B_1}{2\gamma_2\left(\frac{p+2\delta}{p}\right)^m}\qquad(2.6)$$

For any complex number θ ,

$$\left|a_{p+2}-\theta a_{p+1}^2\right|\le\frac{(p-\mu)B_1}{2\gamma_2\left(\frac{p+2\delta}{p}\right)^m}\max\left\{1,\left|\frac{(p-\mu)\left(2\gamma_2\theta\left(\frac{p+2\delta}{p}\right)^m-\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}\right)}{\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}}B_1-\frac{B_2}{B_1}\right|\right\}.\qquad(2.7)$$

The results are sharp.

Proof. If $f(z)\in S_{1,p,\lambda,\mu,\eta,\delta}^m(\phi)$, then there is a Schwarz function

$$w(z)=w_1z+w_2z^2+\cdots\ \in\Omega$$

such that

$$\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)}=\phi\big(w(z)\big)\qquad(2.8)$$

since

$$\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}f(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)} = 1 + \frac{\gamma_1}{(p-\mu)}\left(\frac{p+\delta}{p}\right)^m a_{p+1}z + \\ + \frac{1}{(p-\mu)}\left[2\gamma_2\left(\frac{p+2\delta}{p}\right)^m a_{p+2} - \gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m} a_{p+1}^2\right]z^2 + \cdots$$

We have from (2.8),

$$a_{p+1} = \frac{(p-\mu)w_1}{\gamma_1\left(\frac{p+\delta}{p}\right)^m} B_1 \quad , \tag{2.10}$$

and

$$a_{p+2} = \frac{(p-\mu)}{2\gamma_2\left(\frac{p+2\delta}{p}\right)^m}\{B_1w_2 + (B_2 + (p-\mu)B_1^2)w_1^2\} \tag{2.11}$$

Therefore, we have

$$a_{p+2} - \theta a_{p+1}^2 = \frac{(p-\mu)B_1}{2\gamma_2\left(\frac{p+2\delta}{p}\right)^m}\{w_2 - v w_1^2\} \tag{2.12}$$

where

$$v := \frac{(p-\mu)B_1\left(2\gamma_2\theta\left(\frac{p+2\delta}{p}\right)^m - \gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}\right)}{\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}} - \frac{B_2}{B_1} \tag{2.13}$$

The results (2.4)-(2.7) are established by an application of Lemma 1.5 and inequality (2.7) by Lemma 1.6.

To show that the bounds in (2.4)-(2.7) are sharp, we define the functions $K_{\phi n}$ $(n = 2,3, \dots)$ by

$$\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}K_{\phi n}(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}K_{\phi n}(z)} = \phi(z^{n-1}), \qquad K_{\phi n}(0) = \left(K_{\phi n}\right)'(0) - 1 = 0$$

and the functions F_r , G_r $(0 \le r \le 1)$ defined by

$$\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}F_r(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}F_r(z)}=\phi\left(\frac{z(z+r)}{1+rz}\right), \qquad F_r(0)=F_r{}'(0)-1=0$$

and

$$\frac{N_{0,z}^{m,\lambda+1,\mu+1,\eta+1,\delta,p}G_r(z)}{N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}G_r(z)}=\phi\left(-\frac{z(z+r)}{1+rz}\right), \qquad G_r(0)=G_r{}'(0)-1=0$$

respectively, it is clear that the functions $K_{\phi n}$, F_r and G_r belong to the class $S_{1,p,\lambda,\mu,\eta,\delta}^{m}(\phi)$. If $\theta<\sigma_1$ or $\theta>\sigma_2$, then the equality holds if and only if f is $K_{\phi 2}$ or one of its rotations. If $\sigma_1<\theta<\sigma_2$, the equality holds if and only if f is $K_{\phi 3}$ or one of its rotations. If $\theta=\sigma_1$, then the equality holds if and only if f is F_r or one of its rotations. If $\theta=\sigma_2$, then the equality holds if and only if f is G_r or one of its rotations.

Theorem 2.2. Let $\lambda\geq 0$; $\mu<p+1$; $\eta>\max(\lambda,\mu)-p-1$, $b\in\mathbb{C}\backslash\{0\}$; $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$; $\delta\geq 0$ and $p\in\mathbb{N}$. Further, let $\phi(z)=1+B_1z+B_2z^2+\cdots$, where B_n□s are real with $B_1>0$, $B_2\geq 0$. If $f(z)$ given by (1.1) belongs to $S_{b,p,\lambda,\mu,\eta,\delta}^{m}(\phi)$, then for any complex number θ, we have

$$|a_{p+2}-\theta a_{p+1}^2|\leq\frac{(p-\mu)|b|B_1}{2\gamma_2\left(\frac{p+2\delta}{p}\right)^m}\max\left\{1,\left|\frac{(p-\mu)b\left(2\gamma_2\theta\left(\frac{p+2\delta}{p}\right)^m-\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}\right)}{\gamma_1^2\left(\frac{p+\delta}{p}\right)^{2m}}B_1-\frac{B_2}{B_1}\right|\right\}. \tag{2.14}$$

The result is sharp.

Proof. The proof is similar to the proof of Theorem 2.1.

References

1- R. M. Ali, V. Ravichandran, and N. Seenivasagan, Coefficient bounds for p-valent functions , Appl. Math. Comput. 187 (2007), 35-46.
2- S. M. Amsheri and V. Zharkova, On coefficient bounds for some subclasses of p-valent functions involving certain fractional derivative operator, Int. J. Math. Anal. 6(7)(2012), 321- 331.
3- M. K. Aouf, A. O. Mostafa, and H. M. Zayed, Some characterizations of integral operators associated with certain classes of *p*-valent functions defined by the

Srivastava–Saigo–Owa fractional differintegral operator, Complex Anal. Oper. Theory 10(2016), 1267–1275.

4- W. C. Ma and D. Minda , A unified treatment of some special classes of univalent functions , in Proceeding of the conference on complex analysis, (Tianjin, 1992), Z. Li, F. Ren, L. Yang, and S. Zhang, Eds., Conf. Proc. Lecture Notes Anal., I, pp. 157-169, International Press, Cambridge, Mass, USA, 1994.
5- F. R. Keogh and E. P. Merkes, A coefficient inequality for certain classes of analytic functions , Proc. Amer. Math. Soc. 20(1)(1969), 8-12 .
6- S. Owa, On the distortion theorems-I, Kyungpook. Math. J. 18(1978), 53-59.
7- V. Ravichandran, Y. Polatoglu, M. Bolcal, and A. Sen, Certain subclasses of starlike and convex functions of complex order, Hacettepe J. Math. Statistics, 34 (2005), 9-15.
8- H. M. Srivastava and S. Owa, (Eds), Univalent functions, fractional calculus, and their applications, Halsted Press/Ellis Horwood Limited/John Wiley and Sons, New York/Chichester/Brisbane/Toronto, 1989.
9- H. M. Srivastava and P.M. Karlsson, Multiple Gaussian hypergeometric series, Halsted Press (Ellis Horwood Limited, Chichester), Wiley, New York/ Chichester/ Brishane/ Toronto, 1985.
10- H. M. Zayed, A. Mohammadein, and M. K. Aouf, Sandwich results of p-valent functions defined by a generalized fractional derivative operator with application to vortex motion, Rev . R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM (2018). https://doi.org/10.1007/s13398-018-0559-z.

	مجـلة الـتربـوي Journal of Educational ISSN: 2011- 421X Arcif Q3	معامل التأثير العربي 1.5 العدد 19

الفهـــــــــــرس