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مجلة علمية محكمة تصدر عن كلية التربية

بجامعة المرقب

العدد السادس والعشرين
يناير 2025م

- المجلة ترحب بما يرد عليها من أبحاث وعلى استعداد لنشرها بعد التحكيم .
- المجلة تحترم كل الاحترام آراء المحكمين وتعمل بمقتضاها .
- كافة الآراء والأفكار المنشورة تعبر عن آراء أصحابها ولا تتحمل المجلة تبعاتها .
- يتحمل الباحث مسؤولية الأمانة العلمية وهو المسؤول عما ينشر له .
- البحوث المقدمة للنشر لا ترد لأصحابها نشرت أو لم تنشر .

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يشترط في البحوث العلمية المقدمة للنشر أن يراعى فيها ما يأتي :

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- يرفق بالبحث تزكية لغوية وفق أنموذج معد .
- تعدل البحوث المقبولة وتصحح وفق ما يراه المحكمون .
- التزام الباحث بالضوابط التي وضعتها المجلة من عدد الصفحات ، ونوع الخط ورقمه ، والفترات الزمنية الممنوحة للتعديل ، وما يستجد من ضوابط تضعها المجلة مستقبلا .

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- يخضع البحث في النشر لأولويات المجلة وسياستها .
- البحوث المنشورة تعبر عن وجهة نظر أصحابها ، ولا تعبر عن وجهة نظر المجلة .

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DIMENSION FUNCTIONS ON TOPOLOGICAL SPACES

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Abstract
This paper investigates the inductive and covering dimension functions in topological spaces, highlighting their significance in the classification and analysis of such spaces. Particular attention is given to zero-dimensional and strongly zero-dimensional spaces, with an emphasis on characterizing their properties and determining the conditions under which these spaces possess an inductive and covering dimension of zero. The study underscores the broader relevance of dimension theory in understanding and organizing the fundamental structure of topological spaces.
Keywords: Clopen set, cover, covering dimension, dimension function, inductive dimension, order, refinment, strongly zero-dimension space, subspace, shrinking, Tychonoff space, zero-sets, zero-dimension space.

ملخص:
هذه الورقة تتحقق من دوال البعد الاستقرائية والغطائية في الفضاءات التبولوجية، مع توضيح أهميتها في تصنيف وتحليل هذه الفضاءات. الاهتمام الخاص في هذه الورقة يكون للفضاءت ذات البعد الصفري والفضاءات الصفرية القوية، مع التركيز على توصيف خصائصها وتحديد الظروف التي تمتلك فيها هذه الفضاءات بعدا استقرائيا وغطائيا مساويا للصفر. تؤكد هذه الدراسة على الأهمية الاوسع لنظرية الابعاد لفهم وتنظيم البنية الأساسية للفضاءات التبولوجية.

1. Introduction

Dimension theory has a rich history, dating back to significant contributions from pioneers like Henri Lebesgue, Pavel Urysohn, and Karl Menger. These mathematicians extended the concept of dimension beyond simple geometric settings, formalizing it for more general topological spaces. Their work established the rigorous foundations of modern dimension theory, introducing essential concepts such as the inductive and covering dimension functions.

Dimension functions are crucial for understanding and classifying topological spaces, particularly zero-dimensional spaces. These are spaces where every point has a basis of clopen sets (sets that are both open and closed). Zero-dimensional spaces play an essential role in topology, offering foundational insights into the structure and behavior of more complex spaces. Moreover, they share a unique and significant relationship with dimension functions, particularly the inductive dimensions ind, Ind, and covering dimension dim, which aid in the classification and analysis of topological spaces.

For any topological space X, three key numerical dimensions can be assigned: the small inductive dimension ind(X), the large inductive dimension Ind(X), and the covering dimension dim(X). Each of these dimensions characterizes spaces in distinct ways. The natural domains for ind, Ind, and dim are regular, normal, and Tychonoff spaces, respectively. These dimensions provide a comprehensive framework for studying the complexity and structure of topological spaces.

The primary focus of this paper is Section 4, which introduces Theorems 4.1, 4.5, 4.8, and 4.10. These theorems provide necessary and sufficient conditions for typical spaces to possess dimension functions equal to zero. By thoroughly analyzing these conditions, the study aims to advance the understanding of dimension functions and their practical relevance. These characterizations highlight the broader importance of dimension functions in classifying topological spaces and reveal the intricate connections between zero-dimensionality and more complex dimensional structures.

2. Definitions and Preliminaries

Dimension functions are various mathematical functions that assign "dimension" to a topological space depending on the structure and properties of the space. These functions aim to capture different aspects of dimensionality. The common dimension functions on topology are the small (ind) and large (Ind) inductive dimensions and the covering dimension (dim).

The dimension function ind(X) of a topological space X defines the "dimension" of the space X by examining the complexity of boundaries of open subsets of the space.

The definition of the small inductive dimension is stated as follows:

Definition 2.1 Let X be a regular space and let ind(X) be an integer larger than or equal to -1 or the "infinite number" ∞. We define ind(X) to be the small inductive dimension function of X *assigned to the space* X *and satisfies the following conditions*:

(1) $ind(X) = -1$ if and only if $X = \phi$;

(2) $ind(X) \leq n$, where $n = 0,1,2,\ldots$, if any point $x \in X$ and any neighborhood $U \subseteq X$ of x there exists an open set $V \subseteq X$ where $x \in V \subseteq U$ and $ind(bd(V)) \leq n - 1$.

(3) ind(X) = n if and only if ind(X) ≤ n and ind(X) > n-1, so that $ind(X) \leq n - 1$ does not hold;

(4) $ind(X) = \infty$ if $ind(X) > n$ for $n = -1,0,1,\ldots$.

The dimension function ind is called the Menger-Urysohn dimension.

The regularity property on a space X is hereditary, therefore if ind(X) is defined on X, it is defined for any subspace of X.

The dimension function ind(X) of a topological space X is the smallest integer n such that every point x ∈ X has arbitrarily small neighborhoods whose boundaries have dimension n−1. If no such n exists, we say ind(X)= ∞.

Let $X = \mathbb{R}^2$, the Euclidean plane, and consider any point $x \epsilon X$. Take a small open ball B(x) centered at x with radius $\epsilon > 0$. The boundary of this open ball is a circle S^1 with 1-dimensional. Since every point in X has neighborhoods whose boundaries are 1-dimensional then ind(X)=2.

Theorem 2.2 If S is any subspace of a regular space X then $ind(S) \leq ind(X)$.

Proof. The theorem is true if $ind(X) = \infty$. Suppose that $ind(X) < \infty$. To use induction for $ind(X)$. If $ind(X) = -1$, then by (MU1), $X = \phi$, which implies $S = \phi$, hence $ind(S) = -1$. Assume that the inequality is proved for all spaces X with $ind(X) \leq n - 1$. Consider a regular space X with $ind(X) = n$, let S be a subspace of X, such that $x \in S$ and a neighborhood G of the point x. Thus there exists an open subset U of X that satisfies $G = S \cap U$. Since $ind(X) \leq n$, so by (MU2) there exists an open set $V \subseteq X$ such that $x \in V \subseteq U$ and $ind(bd(V)) \leq n - 1$. The set $W = S \cap V$ is open in S and satisfies $x \in W \subseteq G$ and $bd_S(W) = S \cap (\overline{S \cap V}) \cap \overline{S \backslash V}$. Thus $bd_S(W)$ is a subspace of $bd(V)$, hence by the inequality assumption $ind(bd_S(V)) \leq n - 1$ which together with (MU2) yields the inequality $ind(S) \leq n$.

The dimension function Ind(X) of a topological space is similar to dimension function ind(X) but is defined by the existence of neighborhoods whose boundaries have dimensions that gradually decrease. Here's how it is defined :

Definition 2.3 Let X be a normal space and let Ind(X) be an integer larger than or equal to -1 or

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the "infinite number" ∞. We define Ind(X) to be the large inductive dimension function of X *assigned to the space* X *and satisfies the following conditions*:

(1) $Ind(X) = -1$ if and only if $X = \phi$;

(2) $Ind(X) \leq n$, where $n = 0,1,2,\ldots$, if any closed set $B \subseteq X$ and any open set $U \subseteq X$ which contains B , there exists an open set $V \subseteq X$ where $B \subseteq V \subseteq U$ and $Ind(bd(V)) \leq n - 1$.

(3) $Ind(X) = n$ if and only if $Ind(X) \leq n$ and $IndX > n - 1$;

(4) $Ind(X) = \infty$ if $Ind(X) > n$ for $n = -1,0,1,\ldots$.

The dimension function Ind is called the Brouwer-Cech dimension.

If $Ind(X)$ is defined then $Ind(F)$ is defined for every closed subspace F of X.

The next theorem is in [6]

Theorem 2.4 If F is a closed subspace of a normal space X then $Ind(F) \leq Ind(X)$.

In the last theorem, when F is any subspace of X but not closed, the theorem does not hold. For examle, consider $\mathbb{R}^2$, the usual Euclidean plane, which is a normal space with Ind($\mathbb{R}^2$)=2. Define $A = \mathbb{R}^2 \setminus \{(x,y) \epsilon \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, the complement of the closed unit disk $D = \{(x,y) \epsilon \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. The set A consists of the Euclidean plane with the interior of the unit disk removed, leaving a space with a hole. Let $F = \{(x,y) \epsilon \mathbb{R}^2 : x^2 + y^2 = r^2, 1 < r \leq 2\}$ be a closed subset of the subspace A. To satisfy Ind(A) ≤ 2, the boundary bd(U) where U is a neighborhood in A must be separable into disjoint closed subsets by removing a subset of dimension ≤ 1. Since bd(U) cannot be separated by removing subsets of dimension ≤1, the condition for Ind(A) ≤ 2 fails. This implies that Ind(A) > 2.

The order of a family: Let $\mathcal{K} = \{K_s\}_{s \in S}$ be a family of subsets of a set X which is the largest integer n such that the family $\mathcal{K}$ contains $n + 1$ sets with non-empty intersection, or the "infinite number" ∞, if no such integer exists. Thus if the order of a family $\mathcal{K}$ equals n then for any $n + 2$ members $K_{s_1}, K_{s_2}, \ldots, K_{s_{n+2}}$ of $\mathcal{K}$ we have $K_{s_1} \cap K_{s_2} \cap \ldots \cap K_{s_{n+2}} = \phi$. The order of a family $\mathcal{K}$ is denoted by $ord(\mathcal{K})$.

Remark that if a family $\mathcal{K}$ has order -1, then it consists of the empty set alone and if it has order 0, then it consists of pairwise disjoint sets not all of which are empty.

A subset M of a space X is called zero-set if there exists a continuous function $f: X \longrightarrow I$ such that $M = f^{-1}(\{0\})$. If M is a zero-set, then M^c is called a cozero-set. The countable intersection of zero-sets is a zero-set and the countable union of cozero-sets is a cozero-set.

If f is a continuous real-valued function on a space X, then $\{x: f(x) \geq r\}$, $\{x: f(x) \leq r\}$ and $\{x: f(x) = r\}$ are zero-sets and the set $\{x: f(x) < r\}$ is a cozero-set for all $r \in \mathbb{R}$.

Before stating the definition of the covering dimension "*dim*", we provide the following definitions:

(i) Let $\mathcal{U}$ be a cover of a space $X. Then$ $\mathcal{U}$ is a refinement of another cover $\mathcal{V}$ of the same space X if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that $U \subseteq V$. Then $ord(U) \leq ord(V)$, and every subcover of V is a refinement of V.

(ii) Let $\mathcal{A} = \{A_i\}_{i \in I}$ *be a cover* of a space X. Then a cover $\mathcal{B} = \{B_i\}_{i \in I}$ of X is a shrinking of $\mathcal{A}$ if $B_i \subseteq A_i$ for all $i \in I$. A shrinking is open (closed) if all its members are open (closed) subsets of the space X.

Note that: Every shrinking $\mathcal{B}$ of a cover $\mathcal{A}$ is a refinement of $\mathcal{A}$ and $ord(\mathcal{B}) \leq ord(\mathcal{A})$.

The covering dimension (dim) of a topological space X is closely linked to cozero-sets and finite refinements. Cozero-sets, which form open covers of X, serve as building blocks for finite refinements. For a space with $dim(X) = n$, every open cover can be refined so that no point is included in more than n+1 sets. By organizing cozero-sets through partitions of unity or continuous functions, they reflect the local topological complexity of X. This connection highlights how dim quantifies the "layered" structure of a space. Cozero-sets help verify whether refinements satisfy dimensional constraints, offering a practical tool for analyzing topological spaces and deepening our understanding of their structure.

This concept of covering dimension is defined as follows:

Definition 2.5 Let X be a Tychonoff space and let dim(X) be an integer larger than or equal to -1 or the "infinite number" ∞. We define dim(X) to be the covering dimension function of $X\ assigned\ to\ the\ space\ X\ and\ satisfies\ the\ following\ conditions$:

(1) $dim(X) = -1$ if and only if $X = \phi$;

(2) $dim(X) \leq n$, if every finite cover by cozero-sets of the space X has a finite refinement consisting of cozero-sets of order $\leq n$;

(3) $dim(X) = n$ if $dim(X) \leq n$ and $dim(X) > n - 1$;

(4) $dim(X) = \infty$ if $dim(X) > n$ for $n = -1,0,1,\ldots$

The covering dimension dim is called the Cech-Lebasgue dimension.

Le S be a subset of a space X . Then S is called C^*-embedded in a space X if every continuous function $f: S \to I$ can be extended to a continuous function $F: X \to I$. This property ensures a strong topological connection between S and X, which plays a role in preserving dimension related properties.

For the proof of the next theorem see [5].

Theorem 2.6 If S is a C^*-embedded subspace of aTychonoff space X, then $dim(S) \leq dim(X)$.

The next two theorems show that finite covers, shrinkings, and cozero-sets are central to characterizing the condition dim(X) ≤ n for a Tychonoff space X. These characterizations connect the concepts of finite covers, shrinkability, and cozero-sets to dimension theory.

Theorem 2.7 Any Tychonoff space X has $dim(X) \leq n$ if and only if any finite cover by cozero-sets of the space X has a shrinking contains cozer-sets of order $\leq n$.

Proof. ($\Longrightarrow$) Let $\{U_i\}_{i=1}^{k}$ be a cover by cozero-sets of a Tychonoff space X satisfying $dim(X) \leq n$. Then there exists a refinement $W = \{W_i\}_{i=1}^{l}$ consists of cozero-sets of order $\leq n$ and for each $j \leq l$ choose an $i(j) \leq k$ such that $W_i \subseteq U_{i(j)}$. Let $V_i = \cup \{W_i : i(j) = i\}$. The collection $\{V_i\}_{i=1}^{k}$ is a shrinking consisting of cozero-sets of the cover $\{U_i\}_{i=1}^{k}$ and has order not exceeding n. Since each point of X is in some members of W and then in some V_i, each point of X is in ant most $n + 1$ members of W. Implies that each of them is associated with a unique U_i. Therefore x is at most in $n + 1$ elements of $\{V_i\}_{i=1}^{k}$.

($\Longleftarrow$) Let $\{U_i\}_{i=1}^{k}$ be a cover by a shrinking $\{V_i\}_{i=1}^{k}$ consisting of cozero-sets of order$\leq n$ and by the definition of shrinking $\{V_i\}_{i=1}^{k}$ is a finite refinement consisting of cozero-sets of order$\leq n$. So $dim(X) \leq n$.

The next theorem was proved in [3].

Theorem 2.8 A Tychonoff space X has $dimX \leq n$ if and only if every finite cover $\{U_i\}_{i=1}^{n+2}$ by cozero-sets of X has a shrinking $\{W_i\}_{i=1}^{n+2}$ consisting of cozero-sets such that $\bigcap_{i=1}^{n+2} W_i = \phi$.

3.Topological spaces with zero-dimensionality property:

The zero-dimensionality property of topological spaces is a fundamental concept in topology, characterized by simplicity in structure and lack of connectedness. A topological space with this property is called a zero-dimensional space and has features such as a basis consisting of clopen (simultaneously open and closed) sets, and a dimension of zero in both the inductive and covering dimension senses. It serves as a basic, concrete example for exploring fundamental topological concepts like dimension, separability, and disconnectedness.

Definition 3.1 A space X is called zero-dimensional if X is a non-empty T_1-space and has a base consisting of clopen sets.

Any set with a discrete topology (where every subset is open) is zero-dimensional. This is because it satisfies the T_1 separation property and has a base of clopen sets.

Let $\mathbb{R}$ be the set of real numbers and $\mathcal{J}$ be the family of all intervals $[x, r)$ where $x, r \in \mathbb{R}$, $x < r$. Then the members of $\mathcal{J}$ are clopen with respect to the topology generated by $\mathcal{J}$ on $\mathbb{R}$. This topology is zero-dimensional space and called "Sorgenfrey-line" and we denote it by $\mathbb{R}_S$.

Proposition 3.2 Every non-empty subspace of a zero-dimensional space is zero-dimensional.

Proof. Since every subspace of a T_1-space is T_1, so zero-dimensionality property will be a hereditary property.

The following proposition presents the natural relationship between Tychonoff spaces and zero-dimensional spaces.

Proposition 3.3 A topological space is Tychonoff if it is zero-dimensional space.

Proof. Let X be a zero-dimensional space. Let $y \in X$ and A be a closed subset of X such that $y \notin A$. Then $y \in A^c = U$, where U is open set in X. By zero-dimensionality of X, there exists a clopen set V in X such that $y \in V \subseteq U$. Let $f: X \longrightarrow I$ be the continuous function defined by f(x)=0, if $x \in V$ and $f(x) = 1, if\ x \notin V$. Then $f(y) = \{0\}$ and $f(x) = \{1\}$ for each $x \in A$. Therefore, X is a Tychonoff space.

The real line $\mathbb{R}$ with the usual topology is an example of a Tychonoff space which is not zero-dimensional. The space $\mathbb{R}$ is a Tychonoff space because it can be separate points from closed sets with continuous functions. However, $\mathbb{R}$ is not zero-dimensional because it is not totally disconnected. A zero-dimensional space is totally disconnected, meaning its connected components are singletons. In $\mathbb{R}$, the connected components are intervals (not single points), which makes it non-zero-dimensional.

Theorem 3.4 The cartesian product $\Pi_{i\in I} X_i$ where $I \neq \phi$ and $X_i \neq \phi$ for all $i \in I$ is a zero-dimensional space if and only if X_i is zero-dimensional for all $i \in I$.

Proof. ($\Longrightarrow$) Let $X = \Pi_{i\in I} X_i$ be a zero-dimensional space. To show that X_i is zero-dimensional for each $i \in I$. For each $i \in I$, X_i is homeomorphic to a subspace of X. Since X is a T_1-space and T_1 is hereditary property so by Proposition 3.2, X_i is zero-dimensional for all $i \in I$.

($\Longleftarrow$) Let $x \in X = \Pi_{i\in I} X_i$ and U be an open set in X with $x \in U$. Let $B = p_{i_1}^{-1}(U_1) \cap p_{i_2}^{-1}(U_2) \cap \ldots \cap p_{i_n}^{-1}(U_n)$ be a basic neighborhood of x such that $x \in B \subseteq U$. Since $x = (x_i)_{i\in I}$ then U_k is an open set containing x_{i_k} for all $k = 1,2,\ldots,n$. Since X_i is zero-dimensional, let W_k be a clopen set in X_{i_k}such that $x_{i_k} \in W_k \subseteq U_k$. Let $W = p_{i_1}^{-1}(W_1) \cap p_{i_2}^{-1}(W_2) \cap \ldots \cap p_{i_n}^{-1}(W_n)$, then W is a copen set in X such that $x_k \in W \subseteq B \subseteq U$. Therefore $x \in W \subseteq U$ and because X is a T_1-space so X is zero-dimensional.

Definition 3.5 A space X is called strongly zero-dimensional if X is a non-empty Tychonoff space and any cover $\{U_i\}_{i=1}^{k}$ of X by cozero-sets has an open refinement $\{V_i\}_{i=1}^{m}$ such that $V_i \cap V_j = \phi$ whenever $i \neq j$.

In the above definition the refinement $\{V_i\}_{i=1}^{m}$ consists of clopen sets and thus is a cover of X by cozero-sets.

A strongly zero-dimensional space is a space where every open set has a clopen base. A subspace of a strongly zero-dimensional space inherits the property of having a basis consisting of clopen sets in the subspace topology. Thus, subspaces of strongly zero-dimensional spaces retain a zero-dimensional structure. However, the subspace of strongly zero-dimension space does not necessarily inherit the property of having a clopen basis for all open sets in the subspace. Therefore, not every subspace of strongly zero-dimensional space is strongly zero-dimensional space.

Let K and H be two subsets of a space $X. Then\ K\ and\ H$ are called completely separated if there exists a continuous function $f: X \longrightarrow I$ such that $f(x) = \{0\}$ for each $x \in K$ and $f(x) = \{1\}$ for each $x \in H$ then we say that f separates the sets K and H.

Proposition 3.6 Let A and B be disjoint zero-sets in the space X. Then A and B are completely separated.

Proof. Let $g, h: X \longrightarrow I$ such that $A = g^{-1}(\{0\})$ and $B = h^{-1}(\{0\})$. Let $f: X \longrightarrow I$ be a function defined by f(x)=g(x)/(g(x)+h(x)) for each $x \in X$. Since $A \cap B = \phi$, hence f is a continuous function and if $x \in A$ then $f(x) = \{0\}$ and if $x \in B$ then $f(x) = \{1\}$.

If A and B are disjoint zero-sets in a space X then $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$ for some continuous function $f: X \longrightarrow I$.

The next theorem is in [5].

Theorem 3.7 Let X be a non-empty Tychonoff space. Then X is strongly zero-dimensional space if and only if for every pair A, B of completely separated subsets of X there exists a clopen set $U \subseteq X$ where $A \subseteq U \subseteq X \backslash B$.

4. Applications of dimension functions to zero-dimensional spaces:

This section presents the main results of the paper, which provide various characterizations of topological spaces where the dimension functions are zero. By exploring these characterizations, we aim to deepen the understanding of the conditions that lead to zero-dimensional spaces exhibiting these dimension properties.

Note that if U is a clopen subset of a space X, then $bd(U) = \overline{U} \cap \overline{U^c} = U \cap U^c = \phi$.

The regularity of a space offers a clear characterization for a space X to have $ind(X) = 0$ along with the zero-dimensionality property. The following theorem provides a concrete interpretation of this concept.

Theorem 4.1 A regular space X has ind(x)=0 if and only if X is zero-dimensional.

Proof. ($\Longrightarrow$) Let X be a regular space with $ind(X) = 0$. Let $x \in X$ and U open in X such that $x \in U$. Since $ind(X) = 0$, there exists open set V in X such that $ind(bd(V)) \leq -1$. So $bd(V) = \phi$ and then V is clopen and $x \in V \subseteq U$. Implies that X has a base $\mathcal{B}$ consisting of clopen sets where $\mathcal{B} = \{V \subseteq X: V \text{ is clopen }\}$. Since $X \neq \phi$ so X is zero-dimensional.
($\Longleftarrow$) Let $x \in X$ and U be open in X, where $x \in U$. Since X has a base $\mathcal{B}$ consisting of clopen sets, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$ and $bd(B) = \phi$. Hence $ind(bd(B)) = -1$ and $ind(X) \leq 0$. Since $X \neq \phi$, so $ind(X) = 0$.

Corollary 4.2 A regular space X has $ind(X) = 0$ if and only if any point in X has a local base consists of clopen sets.

Proposition 4.3 Every non-empty countable metrizable space X is zero-dimensional.

Proof. Let x be any point of a countable metrizable space X and $\{x_1, x_2, x_3, \dots\}$ be the set of other points in X. Let d be a metric on X and $s_i = d(x, x_i)$ for all $i \in \mathbb{N}$. The set $\{s_1, s_2, s_3, \dots\}$ is a countable subset of $\mathbb{R}$. If $B_d(x, \epsilon)$ is any spherical neighborhood of x., then there exists a positive real number δ, $\delta < \epsilon$ and $\delta \neq s_i$ for all $i \in \mathbb{N}$. Thus $B_d(x, \delta) \subseteq B_d(x, \epsilon)$ and there are no points of X at a distance δ from x. Hence the set $bd(B_d(x, \delta)) = \phi$ and $B_d(x, \delta)$ is a clopen set. This means X has a local base consisting of clopen sets. Since x was any point in X so by Corollary 4.2, X is zero-dimansional.

Corollary 4.4 The set of rational numbers with the usual topology is zero-dimensional.

Note that not every metrizable space X is zero-dimensional. For example, the set $\mathbb{R}$ with the usual topology is metrizable but not zero-dimensional.
The following theorem shows that the compactness of a topological space offers a characterization that allows a space X to have $Ind(X) = 0$, in addition to possessing the zero-dimensionality property. This result can be found in Nagata's work on compact Hausdorff spaces and the theory of dimension see [7].

Theorem 4.5 Let X be a compact Housdorff space. Then X has Ind(X)=0 if and only if X is zero-dimensional.

Proof. ($\Longrightarrow$) Let X be a compact Hausdorff space with $Ind(X) = 0$. Let $x \in X$ and U be an open subset of X containing x. So $\{x\}$ is a closed set in X and $\{x\} \subseteq U$ and $Ind(bd(V)) \leq -1$. Implies that $bd(V) = \phi$ and then V is a clopen set. Therefore, X has a base $\mathcal{B}$ consisting of clopen sets where $\mathcal{B} = \{V: V \text{ is open in } X\}$. Since $X \neq \phi$, hence X is zero-dimension.
($\Longleftarrow$) Let X be a compact Hausdorff space and E be a closed subset of X. Let U be an open set with $E \subseteq U$. By zero-dimensionality of X, there exists a clopen set V_x in X and for each $x \in E$, we have $x \in V_x \subseteq U$ and $E \subseteq (\bigcup_{x \in E} V_x) \subseteq U$. Since E is a compact set , there exist $V_{x_1}, V_{x_2}, \dots V x_n$ such that $E \subseteq (\bigcup_{i=1}^{n} V_{x_i}) \subseteq U$ where $\bigcup_{i=1}^{n} V_{x_i}$ is a clopen set in a normal space X. Thus $Ind(bd(\bigcup_{i=1}^{n} V_{x_i}) = -1$ which is mean that $Ind(X) \leq 0$. Since $X \neq \phi$, therefore $Ind(X) = 0$.

Theorem 4.6 A normal space X has $Ind(X) = 0$ if and only if any disjoint pair of closed sets can be separated by clopen sets.

Proof. ($\Longrightarrow$) Let X be a normal space with $Ind(X) = 0$. Let E, F be disjoint closed sets in X. Since $E \subseteq F^c$ and $Ind(X) = 0$ hence there exists a clopen set V such that $E \subseteq V \subseteq F^c$. Because $V \subseteq F^c$ so $F \subseteq V^c$ and then V, V^c are disjoint clopen sets such that $E \subseteq V$ and $F \subseteq V^c$.

($\Longleftarrow$) Let E be closed and U be open subsets of a normal space X with $E \subseteq U$. So $E \cap U^c = \phi$. Then E, U^c are disjoint closed sets and by our hypothesis there exist V, W which are clopen in X such that $E \subseteq V, U^c \subseteq W$ and $V \cap W = \phi$ and $V \cup W = X$, $bd(V) = \phi$. Therefore, $V = W^c \subseteq U$ and then $E \subseteq V \subseteq U$. Thus $Ind(bd(V)) = -1$ and then $Ind(X) \leq 0$. Since $X \neq \phi$ so $Ind(X) = 0$.

By a swelling of a family $\{A_i\}_{i\in I}$ of a space X we mean any family $\{B_i\}_{i\in I}$ of subsets of the space X such that $A_i \subseteq B_i$ for all $i \in I$ and for any finite set of indices $i_1, i_1, \ldots, i_n \in I$ we have $A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_n} = \phi$ if and only if $B_{i_1} \cap B_{i_2} \cap \ldots \cap B_{i_n} = \phi$.

Note that swelling is open if its members are open subsets of the space X and every swelling $\mathcal{B}$ of a family $\mathcal{A}$ satisfies the equality $ord(\mathcal{B}) = ord(\mathcal{A})$.

Remark that a finite family consists of zero-sets of a space X has a swelling consisting of cozero-sets.

Proposition 4.7 Every finite cover by cozero-sets of a space X has a shrinking consists of zero-sets.

Proof. Let $\mathcal{U} = \{U_i\}_{i=1}^{k}$ be a cover of cozero-sets of the space X. The family $\{X\backslash U_i\}_{i=1}^{k}$ consists of zero-sets with empty intersection because if $x \in X$ and $x \in \cap \{X\backslash U_i\}_{i=1}^{k}$ then $x \in X$ and $x \notin U_i$ for each $i = 1,2,\ldots,k$, which gives a contradiction. The family $\{X\backslash U_i\}_{i=1}^{k}$ has a swelling $\mathcal{F} = \{V_i\}_{i=1}^{k}$ consisting of cozero-sets. Then the family $\mathcal{F}$ is a stringing of $\mathcal{U}$ consisting of zero-sets.

In a normal space X, dim(X)=0 is equivalent to Ind(X)=0, both measuring the "degree of separation" within the space. dim(X)=0 means every point has a clopen neighborhood base, while Ind(X)=0 means closed sets can be separated by open sets in line with the inductive dimension. This equivalence shows that in normal spaces, topological and inductive dimensions coincide when the space is zero-dimensional as stated in the next theorem (see [4]).

Theorem 4.8 A normal space X has $dim(X) = 0$ if and only if $Ind(X) = 0$.

Proof. ($\Longrightarrow$) Suppose $dim(X) = 0$, to show that $Ind(X) = 0$. Let E be closed in X and U be open in X with $E \subseteq U$. Then E, U^c are disjoint closed sets in X. Then there exists a continuous function $f: X \longrightarrow I$ such that $f(E) = \{0\}, f(U^c) = \{1\}$. Let $W_1 = f^{-1}((0,1]), W_2 = f^{-1}([0,1))$, then $\{W_1, W_2\}$ is a cover of X by cozero-sets and since $dim(X) = 0$, so by Theorem 2.8, there exists a shrinking $\{A, B\}$ consisting of cozero-sets such that $A \subseteq W_1, B \subseteq W_2$ and $A \cap B = \phi$. Implies that A, B are clopen sets in X and $E \subseteq B \subseteq W_2$ such that $Ind(bd(B)) = -1$. Thus $Ind(X) \leq 0$. since $X \neq \phi$ so $Ind(X) = 0$.

(⟸) Let X be a normal space a normal space where $Ind(X) = 0$. Let $\{U_1, U_2\}$ be an open cover of X by cozero-sets. If $U_2 \neq \phi$, let $F = U^c$. Then F is a cozero-set in X and $F \subseteq U_1$. Since $Ind(X) = 0$, so there exists a clopen set V_1 such that $F \subseteq V_1 \subseteq U_1$. Let $V_2 = V_1^{v}$,, then $\{V_1, V_2\}$ is a cover of X by clopen sets and $V_1 \cap V_2 = \phi$ such that $V_1 \subseteq U_1$ and $V_2 \subseteq F^c = U_2$. Therefore $V_2 \subseteq U_2$ and then $dim(X) \leq 0$. Since $X \neq \phi$, so $dim(X) = 0$.

The next proposition shows that the strongly zero-dimensional spaces are sub-classes of zero-dimensional spaces.

Proposition 4.9 Every strongly zero-dimensional space X is zero-dimensional.

Proof. Let $x \in X$ and an open subset U of X containing x. Let $f: X \longrightarrow I$ be a continuous function defined by $f(U^c) = \{0\}$ and $f(x) = \{1\}$. Let $A = f^{-1}(\{1\})$ and $B = f^{-1}(\{0\})$. Therefore A and B are two disjoint zero-sets and by Propoosition 3.7, A and B are completely separated sets. Since X is strongly zero-dimensional, hence there exists a clopen set V such that $A \subseteq V \subseteq X \backslash B = U$. Thus $V \subseteq U$ and then $x \in V \subseteq U$. Since $bd(V) = \phi$, hence $ind(bd(V)) = -1$ and $indX \leq 0$. Because $X \neq \phi$, so $indX = 0$ and then X is zero-dimensional.

Remark that not every zero-dimensional space is strongly zero-dimensional. For examle, the Sorgenfrey line $\mathbb{R}_S$ defined above is zero-dimensional but is not strongly zero-dimensional. Because there are open covers in this topology that cannot be refined into a finite collection of clopen sets. For example, the open cover {[x,r): x, r ∈ ℝ and x < r} does not admit a finite refinement of clopen sets, since the open sets are not "small enough" to be separated by a finite number of clopen sets.

The comming theorem demonstrates the equivalence of dim(X)=0 and strong zero-dimensionality in a Tychonoff space, highlighting a key connection between dimension theory and separation properties. In particular, a space X is strongly zero-dimensional if and only if dim(X)=0, which implies that any two disjoint closed sets in X can be separated by a clopen set. This equivalence is guaranteed in Tychonoff spaces, where the separation axioms ensure the presence of sufficient clopen sets. It emphasizes that the zero-dimensionality of a space is intrinsically linked to its capacity to separate disjoint closed sets with clopen sets, establishing its importance as a fundamental concept in topology.

The statement of the next theorem might be found in many difrent works (see [6] and [7]) dealing with the finer distinctions between various classes of zero-dimensional spaces, like in Nagata's research or other foundational texts in dimension theory, which explore the relationships between inductive dimension and zero-dimensionality.

Theorem 4.10 A space X has dim(X) = 0 if and only if X is strongly zero-dimensional.

Proof. (⟹) Suppose $dim(X) = 0$. Let $\{U_1, U_2\}$ be a cover of X by cozero-sets. By Proposition 4.7, the cover has a shrinking $\{W_1, W_1\}$ consisting of cozero-sets such that $W_1 \cap W_2 = \phi$. Therefore X is strongly zero-dimensional.

(⟸) Let X be a strongly zero-dimensional space and let $\mathcal{U} = \{U_i\}_{i=1}^{k}$ be a cover of X by cozero-sets. By Definition 2.5, the cover $\mathcal{U}$ has a finite open refinement $\mathcal{V} = \{V_i\}_{i=1}^{k}$ consisting of cozero-sets and $V_i \cap V_j = \phi$ whenever $i \neq j$ and $1 \leq I, j \leq m$. Thus $ord\mathcal{V} \leq 0$ and by Definition 2.5, we have $dimX \leq 0$. since $X \neq \phi$, hence $\dim(X) = 0$.

Every strongly zero-dimensional space is normal because the strongly zero-dimensional property ensures that any two closed sets with empty intersection can be separated by a clopen set. This separation condition is stronger than the standard separation condition required for normal spaces. However, the converse is not true. For instance, the real line with the usual topology is a normal space but fails to be strongly zero-dimensional.

The following corollary follows from the last theorem and the previous discussion.

Corollary 4.11 A normal space X has $Ind(X) = 0$ if and only if X is strongly zero-dimensional.

Proposition 4.12 Let X be a strongly zero-dimensional space and Y be a C^*-embedded subspace of X. Then Y is strongly zero-dimensional.

Proof. Since X is strongly zero-dimensional, hence by Theorem 4.10, we have $dim(X) = 0$ and by Theorem 2.6, we have $dim(Y) \leq 0$. Since $Y \neq \phi$, hence $dim(Y) = 0$ and therefore Y is strongly zero-dimensional subspace of X.

Conclusion

This study has presented characterizations of zero-dimensional spaces in terms of their inductive and covering dimensions, focusing on the conditions under which these dimensions equal zero. Defined by their clopen bases, zero-dimensional spaces are fundamental in topology, and their relationship with dimension functions provides critical insights into the classification and structural behavior of topological spaces. This paper examined the structural properties that result in zero values for the dimension functions ind, Ind, and dim. This study has made connections between various topological concepts, enhancing the understanding of these unique spaces. Additionally, the study highlighted the equivalences and distinctions among dimension functions in zero-dimensional and strongly zero-dimensional spaces, contributing to a more unified perspective within dimension theory.

We recommend that future research explore the behaviour of dimension functions in broader contexts, providing deeper insights into their properties and enhancing our understanding of their operation across diverse topological settings.

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