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العدد 20

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مجلة التربوي

مجلة علمية محكمة تصدر عن كلية التربية

جامعة المرقب

العدد العشرون

يناير 2022م

هيئـــة تحريـر

مجلة التربوي

- المجلة ترحب بما يرد عليها من أبحاث وعلى استعداد لنشرها بعد التحكيم .
- المجلة تحترم كل الاحترام آراء المحكمين وتعمل بمقتضاها .
- كافة الآراء والأفكار المنشورة تعبر عن آراء أصحابها ولا تتحمل المجلة تبعاتها .
- يتحمل الباحث مسؤولية الأمانة العلمية وهو المسؤول عما ينشر له .
- البحوث المقدمة للنشر لا ترد لأصحابها نشرت أو لم تنشر .

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ضوابط النشر:

يشترط في البحوث العلمية المقدمة للنشر أن يراعى فيها ما يأتي :

- أصول البحث العلمي وقواعده .
- ألا تكون المادة العلمية قد سبق نشرها أو كانت جزءا من رسالة علمية .
- يرفق بالبحث تزكية لغوية وفق أنموذج معد .
- تعدل البحوث المقبولة وتصحح وفق ما يراه المحكمون .
- التزام الباحث بالضوابط التي وضعتها المجلة من عدد الصفحات ، ونوع الخط ورقمه ، والفترات الزمنية الممنوحة للتعديل ، وما يستجد من ضوابط تضعها المجلة مستقبلا .

تنبيهات :

- للمجلة الحق في تعديل البحث أو طلب تعديله أو رفضه .
- يخضع البحث في النشر لأولويات المجلة وسياستها .
- البحوث المنشورة تعبر عن وجهة نظر أصحابها ، ولا تعبر عن وجهة نظر المجلة .

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Sufficient Conditions of Bounded Radius Rotations for Two Integral Operators

Defined by q-Analogue of Ruscheweyh Operator

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Abstract

Motivating by q- analogue theory, in this article, we derive the q-analogue of Ruscheweyh differential Operator using q-hypergeometric function and using this new operator to define integral operators which generalizes many operators introduced before. We also consider supclasses of analytic functions with bounded radius and bounded boundary rotations and study the mapping properties of these classes under these q-intergral operators.

1 Introduction

Let A be the class of all functions of the following form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

Which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in A$ is said to be spiral - like if there exists a real number $\lambda(|\lambda| < \pi/2)$ such that

$$Re\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\} > 0 \ , \qquad (z \in U).$$

The class of all spiral - like functions has been introduced by Spacek [1] in 1933 and we denote it by S_λ^* . Later in 1969, Robertson [2] considered the class $C_\lambda$of analytic functions in U for which $zf'(z) \in S_\lambda^*$.

Let $P_k^\lambda(\xi)$be the class of functions $p(z)$ analytic in U with $p(0)$ =1 and

$$\int_0^{2\pi}\left|\frac{Re\, e^{i\lambda}p(z)-\ \xi cos\lambda}{1-\xi}\right| d\theta \le k\pi\, cos\lambda\,, \qquad z=\ re^{i\theta},$$

where $k\ge 2\,,0\le\xi<1\,,\lambda$ is real with $|\lambda|<\pi/2$.

For $\lambda=0$,this class was introduces in [3] and for $\xi=0$, refer to [4]. For $k=2\,,\lambda=0$ and $\xi\ =0$, the class $P_k^{\lambda}(\xi)$ reduces to the class P of functions $p(z)$ analytic in U with $p(0)$=1 and whose real part is positive. For $k<2$, the functions in p_k may not have positive real part.

One of the most important summation formulas for hypergeometric series is given by the binomial theorem:

$$ {}_2F_1(a\,,c,c;z)=\ \ {}_1F_0(a,-;z)$$
$$=\sum_{k=0}^{\infty}\frac{(a_k)}{k!}z^k=(1-z)^{-a}\,, \qquad a\ \in\mathbb{R},$$

where ${}_2F_1$ denotes Gaussian hypergeometric function and $|z|<1$. The q- analogue of this formula is defined by

$$ {}_1\emptyset_0(a,-\,;q;z)=\sum_{k=0}^{\infty}\frac{(a\,;q)_k}{(q\,;q)_k}\,,|z|<1\,,|q|<1, \qquad (1.1)$$

which was derived by Cauchy (1843), Heine (1847) and by other mathematicians.$(a\,,q)_k$ is the q- analogue of the Pochhammer symbol $(a)_k$ defined by

$$(a\,,q)_k=\begin{cases}1, & k=0:\\ (1-a)(1-aq)(1-aq^2)\dots(1-aq^{k-1}), & k\in\mathbb{N}.\end{cases}$$

It is clear that

$$\lim_{q\to 1}\frac{(q^a;q)_k}{(1-q)^k}=(a)_k.$$

By using the ratio test, one recognize that, if $|q|<1$, the q series (1.1) (also called basic hypergeometric series) converges absolutely for $|z|<1$. For more details concerning the q-theory the reader may refer to ([5],[6]).

Replacing a by$(\delta+1, \delta>-1)$ in (1.1), we now define the q-analogue of Ruscheweyh differential operator $D_q^\delta f : A \rightarrow A$as follows:

$$D_q^\delta f(z)=z\ {}_1\emptyset_0(\delta+1,-;q;z) * f(z)$$

$$=z+\sum_{k=2}^{\infty}\frac{(\delta+1\,;q)_{k-1}}{(q\,;q)_{k-1}}a_k z^k\,, \delta>-1\,, |z|<1\,, |q|<1. \quad (1.2)$$

Observe that if $\delta+1= q^{\mu+1}, \mu>-1$, we have

$$\lim_{q\to 1} D_q^\delta f(z)=z+\lim_{q\to 1}\left[\sum_{k=2}^{\infty}\frac{(q^{\mu+1};q)_{k-1}}{(q\,;q)_{k-1}}a_k z^k\right]$$

$$=z+\sum_{k=2}^{\infty}\frac{(\mu+1)_{k-1}}{(k-1)!}a_k z^k=D^\mu f(z),$$

where $D^\mu f(z)$ is Ruscheweyh differential operator defined in [7].

Next, we define a new subclasses of analytic functions of complex order involving the q- analogue of Ruscheweyh differential operator.

Definition 1.1 *A* function $f\in A$ is said to belong to the class $R_k^\lambda(\xi,b,\delta,q)$ if and only if

$$1+\frac{1}{b}\left(\frac{z\left(D_q^\delta f(z)\right)'}{D_q^\delta f(z)}-1\right)\in P_k^\lambda(\xi),$$

where $k\geq 2\,, 0\leq\xi<1$, λ is real with $|\lambda|<\pi/2, b\in\mathbb{C}-\{0\}$.

Definition 1.2 *A* function $f\in A$ is said to belong to the class $V_k^\lambda(\xi,b,\delta,q)$ if and only if

$$1+\frac{1}{b}\frac{z\left(D_q^\delta f(z)\right)''}{\left(D_q^\delta f(z)\right)'}\in P_k^\lambda(\xi),$$

where $k\geq 2\,, 0\leq\xi<1\,, \lambda$ is real with $|\lambda|<\pi/2, b\in\mathbb{C}-\{0\}$.

Remark 1.1 i) *Letting $\delta=q-1, b=1$, we obtain the classes $R_k^\lambda(\xi)$, $V_k^\lambda(\xi)$ respectively, introduced and studies by Noor et al. [8] and Moulis [9].*

ii) *For* $\delta = q-1$ *and* $\lambda = 0$ *, we have the classes* $R_k(\xi, b), V_k(\xi, b)$, *respectively, introduced and studies by Noor et al. [10].*

iii) *For* $\delta = q-1, k = 2$ *and* $\lambda = 0$ *, we have the classes* $S^*_\xi(b), C_\xi(b)$ *, respectively introduced and studies by Frasion [11].*

iv) *For* $\delta = q-1, b = 1, \xi = 0$ *and* $\lambda = 0$*, we obtain the well known classes* R_k, V_k *of analytic functions with bounded radius and bounded boundary rotations introduced and studies by Tammi [12]and Paatero*[13].

Definition 1.3 *Let* $m \in \mathbb{N} \cup \{0\}, j \in \{1, 2, \dots, m\}$ *, and* $\alpha_j > 0$. *Ones defines the integral operator* $I(f_1, f_2, \dots f_m): A^m \to A$ *as* $I(f_1, \dots f_m) = F$,

$$D_q^\delta F(z) = \int_0^z \left(\frac{D_q^\delta f_1(t)}{t}\right)^{\alpha_1} \dots \left(\frac{D_q^\delta f_m(t)}{t}\right)^{\alpha_m} dt \ (z \in U), \quad (1.3)$$

where $f_j \in A$.

Remark 1.2 *The integral operator* $D_q^\delta F$ *generalizes many operators which were introduced and studied recently.*

i) *For* $\delta + 1 = q^{\mu+1}, \mu > -1,$ *and* $q \to 1$ *, we obtain the integral operator*

$$D^\mu F(z) = \int_0^z \left(\frac{D^\mu f_1(t)}{t}\right)^{\alpha_1} \dots \left(\frac{D^\mu f_m(t)}{t}\right)^{\alpha_m} dt, \quad (1.4)$$

where D^μ is the *Ruscheweyh differential operator. The integral operator*$D^\mu F(z)$ *introduced and studied by G. Oros et al. [14] .*

ii) *For* $\delta = q-1$, $D_q^{q-1} f_j(z) = f_j \in A, j \in \{1, 2, \dots, m\}$ *, we have the integral operator*

$$F_m(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \dots \left(\frac{f_m(t)}{t}\right)^{\alpha_m} dt \quad (1.5)$$

introduced by D. Breaz and N. Breaz [15].

iii) *For* $\delta = q-1\,, m=1, \alpha_1 = \alpha \in [0.1], \alpha_2 = \cdots = \alpha_m = 0$ *and* $D_q^{q-1}f_1 = f \in S^*$,*(consists of functions that are analytic, univalent and starlike),we have the integral operator*

$$F_\alpha(z) = \int_0^z \left(\frac{f\ \ (t)}{t}\right)^\alpha \ dt \qquad (1.6)$$

introduced by Miller et. al. [16].

iv) *For* $\delta = q-1, m=1, \alpha_1 = 1, \alpha_2 = \cdots = \alpha_m = 0$ *and* $D_q^{q-1}f_1 = f \in A$, *we have the integral operator*

$$F(z) = \int_0^z \frac{f\ \ (t)}{t}\ dt \qquad (1.7)$$

introduced by Alexander. [17].

Definition 1.4 *Let* $m \in \mathbb{N}_0, j \in \{1,2,\ldots,m\}$, *and* $\alpha_j > 0$. *Ones defines the integral operator* $I(f_1, f_2, \ldots f_m)\colon A^m \to A$ *as* $I(f_1, \ldots f_m) = H$,

$$D_q^\delta H(z) = \int_0^z \left[\left(D_q^\delta f_1(t)\right)'\ \right]^{\alpha_1} \ldots \left[\left(D_q^\delta f_m(t)\right)'\ \right]^{\alpha_m} dt\ (z \in U), \qquad (1.8)$$

where $f_j \in A$.

Remark 1.3 *The integral operator* $D_q^\delta H$ *generalizes many operators which were introduced and studied recently.*

i) *For* $\delta = q-1$*and* $D_q^{q-1}f_j = f_j \in A, j \in \{1,2,\ldots,m\}$, *we have the integral operator*

$$H(z) = \int_0^z \left(f_1'(t)\right)^{\alpha_1} \ \ldots \left(f_m'(t)\right)^{\alpha_m} \ dt \qquad (1.9)$$

introduced by Breaz et. al. [18].

ii) *For* $\delta = q-1\,, m=1, \alpha_1 = \alpha \in \mathbb{C}, \alpha_2 = \cdots = \alpha_m = 0$ *and* $D_q^{q-1}f_1 = f \in A$, *we have the integral operator*

$$H_\alpha(z) = \int_0^z \left(f_1'(t)\right)^\alpha \, dt \qquad (1.10)$$

introduced by Pfaltzgraff [19] (see also Pascu and pascar [20]).

In this paper, we investigate some properties of the above integral operators $D_q^\delta F$ and $D_q^\delta H$ for the classes $R_k^\lambda(\xi_j, b, \delta, q)$ and $V_k^\lambda(\xi_j, b, \delta, q)$.

2 Main results

Theoem 2.1 *Let* $f_j \in R_k^\lambda(\xi_j, b, \delta, q)$,*for* $j \in \{1, 2, \dots, m\}$ *with* $0 \le \xi_j < 1, b \in \mathbb{C} - \{0\}$. Also let λ be real *with* $|\lambda| < \pi/2, \alpha_j > 0.$ $j \in \{1, 2, \dots, m\}$. *If*

$$0 \le 1 + \sum_{j=1}^{m} \alpha_j(\xi_j - 1) < 1,$$

then the integral operator F defined by (1.3) is in the class $V_k^\lambda(\gamma, b, \delta, q)$ *wi th*

$$\gamma = 1 + \sum_{j=1}^{m} \alpha_j(\xi_j - 1) \qquad (2.1)$$

Proof. Since $f_j \in A$, $j \in \{1, 2, \dots, m\}$, by (1.2), we have

$$\frac{D_q^\delta f_j(z)}{z} = 1 + \sum_{k=2}^{\infty} \frac{(\delta+1\,;q)_{k-1}}{(q;q)_{k-1}} \alpha_{kj} z^{k-1} \neq 0 \quad \text{for all} \quad z \in U.$$

By (1.3), we get

$$\left(D_q^\delta F(z)\right)' = \left(\frac{D_q^\delta f_1(z)}{z}\right)^{\alpha_1} \dots \left(\frac{D_q^\delta f_m(z)}{z}\right)^{\alpha_m}.$$

This equality implies that

$$ln\left(D_q^\delta F(z)\right)' = \alpha_1\left[ln\, D_q^\delta f_1(z) - ln\, z\right] + \cdots + \alpha_m\left[ln\, D_q^\delta f_m(z) - ln\, z\right].$$

By differentiating the last equality, we have

$$\frac{\left(D_q^\delta F(z)\right)''}{\left(D_q^\delta F(z)\right)'} = \sum_{j=1}^{m} \alpha_j\left(\frac{\left(D_q^\delta f_j(z)\right)'}{D_q^\delta f_j(z)} - \frac{1}{z}\right).$$

Multiplying with z and $1/b$ both sides of the last relation we obtain

$$\frac{1}{b}\frac{z\left(D_q^\delta F(z)\right)''}{\left(D_q^\delta F(z)\right)'} = \sum_{j=1}^{m} \alpha_j \frac{1}{b}\left(\frac{z\left(D_q^\delta f_j(z)\right)'}{D_q^\delta f_j(z)} - 1\right)$$

$$= \sum_{j=1}^{m} \alpha_j\left[1 + \frac{1}{b}\left(\frac{z\left(D_q^\delta f_j(z)\right)'}{D_q^\delta f_j(z)} - 1\right)\right] - \sum_{j=1}^{m} \alpha_j\,,$$

or equivalently

$$e^{i\lambda}\left(1 + \frac{1}{b}\frac{z\left(D_q^\delta F(z)\right)''}{\left(D_q^\delta F(z)\right)'}\right) = \left(1 - \sum_{j=1}^{\infty} \alpha_j\right)e^{i\lambda} + \sum_{j=1}^{\infty} \alpha_j\, e^{i\lambda}\left[1 + \frac{1}{b}\left(\frac{z\left(D_q^\delta f_j(z)\right)'}{D_q^\delta f_j(z)} - 1\right)\right].$$

Subtracting and adding $\cos\lambda \sum_{j=1}^{\infty} \alpha_j \xi_j$ on the left hand side and taking the real part, we obtain

$$Re\left\{e^{i\lambda}\left(1 + \frac{1}{b}\frac{z\left(D_q^\delta F(z)\right)''}{\left(D_q^\delta F(z)\right)'}\right) - \gamma cos\lambda\right\} = \sum_{j=1}^{\infty} \alpha_j Re\left\{e^{i\lambda}\left[1 + \frac{1}{b}\left(\frac{z\left(D_q^\delta f_j(z)\right)'}{D_q^\delta f_j(z)} - 1\right)\right] - \xi_j cos\lambda\right\},$$

where γ is given by (2.1). Integrating the last equation, we have

$$\int_0^{2\pi}\left|Re\left\{e^{i\lambda}\left(1 + \frac{1}{b}\frac{z\left(D_q^\delta F(z)\right)''}{\left(D_q^\delta F(z)\right)'}\right) - \gamma cos\lambda\right\}\right| d\theta$$

$$\leq \sum_{j=1}^{\infty} \alpha_j \int_0^{2\pi} \left| Re \left\{ e^{i\lambda} \left[1 + \frac{1}{b} \left(\frac{z\left(D_q^{\delta} f_j(z)\right)'}{D_q^{\delta} f_j(z)} - 1 \right) \right] - \xi_j cos\lambda \right\} \right| d\theta \qquad (2.2)$$

Since $f_j \in R_k^{\lambda}(\xi_j, b, \delta, q), j \in \{1, 2, \dots, m\}$, we have

$$\int_0^{2\pi} \left| Re \left\{ e^{i\lambda} \left[1 + \frac{1}{b} \left(\frac{z\left(D_q^{\delta} f_j(z)\right)'}{D_q^{\delta} f_j(z)} - 1 \right) \right] - \xi_j cos\lambda \right\} \right| d\theta \leq (1 - \xi_j)\, k\pi cos\lambda, \qquad for\ (1 \leq j \leq m) \quad (2.3)$$

Applying (2.4) in (2.3), we conclude

$$\int_0^{2\pi} \left| Re \left\{ e^{i\lambda} \left(1 + \frac{1}{b} \frac{z\left(D_q^{\delta} F(z)\right)''}{\left(D_q^{\delta} F(z)\right)'} \right) - \gamma cos\lambda \right\} \right| d\theta \leq k\pi cos\lambda \sum_{j=1}^{m} \alpha_j (1 - \xi_j)$$

Subsequently, $F \in V_k^{\lambda}(\gamma, b, \delta, q)$ with γ is given by (2.1).

By setting $\lambda = 0$, $\delta = q - 1$ in Theorem 2.1, we obtain the following result.

Corollary 2.1 Let $f_j \in R_k\,(\xi_j, b)\ for\ j \in \{1, 2, \dots, m\}$ with $(0 \leq \xi_j < 1), b \in \mathbb{C} - \{0\}$. Also let $\alpha_j > 0$ $(j \in \{1,2, \dots .. m\})$.If

$$0 \leq 1 + \sum_{j=1}^{m} \alpha_j (\xi_j - 1) < 1, \qquad (2.4)$$

then the integral operator F_m defined by (1.5) is in the class V_k (γ, b) with γ defined as (2.1).

Theorem 2.2 Let $f_j \in V_k^{\lambda}(\xi_j, b, \delta, q)\ for\ j \in \{1, 2, \dots, m\}$ with $(0 \leq \xi_j < 1), b \in \mathbb{C} - \{0\})$ .Also let λ be real with $|\lambda| < \pi/2, \alpha_j > 0, j \in \{1, 2, \dots, m\}$.*If the condition (2.4) satisfied,.then the integral operator H defined by (1.8)is in the class $V_k^{\lambda}(\xi_j, b, \delta, q)$ with γ is defined by (2.1).*

Proof. By (1.8), we get

$$\left(D_q^\delta H(z)\right)' = \left[\left(D_q^\delta f_1(z)\right)'\right]^{\alpha_1} \dots \left[\left(D_q^\delta f_m(z)\right)'\right]^{\alpha_m}$$

This equality implies that

$$\left(D_q^\delta H(z)\right)'' = \sum_{j=1}^{m} \alpha_j \left[\left(D_q^\delta f_j(z)\right)'\right]^{\alpha_j} \frac{\left(D_q^\delta f_j(z)\right)''}{\left(D_q^\delta f_j(z)\right)'} \times \prod_{\substack{r=1 \\ r\neq j}}^{m} \left[\left(D_q^\delta f_r(z)\right)'\right]^{\alpha_r}.$$

By differentiating the last equality and multiplying by z/b , we have

$$\frac{1}{b}\frac{z\left(D_q^\delta H(z)\right)''}{\left(D_q^\delta H(z)\right)'} = \sum_{j=1}^{m} \alpha_j \frac{1}{b}\frac{z\left(D_q^\delta f_j(z)\right)''}{\left(D_q^\delta f_j(z)\right)'}$$

$$= \sum_{j=1}^{m} \alpha_j \left(1 + \frac{1}{b}\frac{z\left(D_q^\delta f_j(z)\right)''}{\left(D_q^\delta f_j(z)\right)'}\right) - \sum_{j=1}^{m} \alpha_j \,,$$

or equivalently

$$e^{i\lambda}\left(1 + \frac{1}{b}\frac{z\left(D_q^\delta H(z)\right)''}{\left(D_q^\delta H(z)\right)'}\right) = \left(1 - \sum_{j=1}^{\infty} \alpha_j\right)e^{i\lambda} + \sum_{j=1}^{\infty} \alpha_j\, e^{i\lambda}\left(1 + \frac{1}{b}\frac{z\left(D_q^\delta f_j(z)\right)''}{\left(D_q^\delta f_j(z)\right)'}\right).$$

Subsequently and adding $\cos\lambda \sum_{j=1}^{\infty} \alpha_j \xi_j$on the left hand side and taking the real part, we obtain

$$Re\left\{e^{i\lambda}\left(1 + \frac{1}{b}\frac{z\left(D_q^\delta H(z)\right)''}{\left(D_q^\delta H(z)\right)'}\right) - \gamma cos\lambda\right\} = \sum_{j=1}^{\infty} \alpha_j Re\left\{e^{i\lambda}\left(1 + + \frac{1}{b}\frac{z\left(D_q^\delta f_j(z)\right)''}{\left(D_q^\delta f_j(z)\right)'}\right) - \xi_j cos\lambda\right\},$$

where γ is given by (2.1). Integrating the last equation, we have

$$\int_0^{2\pi}\left|Re\left\{e^{i\lambda}\left(1 + + \frac{1}{b}\frac{z\left(D_q^\delta H(z)\right)''}{\left(D_q^\delta H(z)\right)'}\right) - \ \gamma cos\right\}\right| d\theta$$

$$\leq \sum_{j=1}^{\infty} \alpha_j \int_0^{2\pi}\left|Re\left\{e^{i\lambda}\left(1 + \frac{1}{b}\left(\frac{z\left(D_q^\delta f_j(z)\right)''}{\left(D_q^\delta f_j(z)\right)'}\right) - \xi_j cos\lambda\right)\right\}\right| d\theta \qquad (2.5)$$

Since $f_j \in R_k^{\lambda}(\xi_j, b, \delta, q), j \in \{1, 2, \dots, m\}$, we have

$$\int_0^{2\pi} \left| Re \left\{ e^{i\lambda} \left(1 + \frac{1}{b} \frac{z\left(D_q^{\delta} f_j(z)\right)''}{\left(D_q^{\delta} f_j(z)\right)'} \right) - \xi_j cos\lambda \right\} \right| d\theta \leq (1 - \xi_j) k\pi cos\lambda, \qquad for(1 \leq j \leq m). \quad (2.6)$$

Applying (2.6) in (2.5), we conclude

$$\int_0^{2\pi} \left| Re \left\{ e^{i\lambda} \left(1 + \frac{1}{b} \frac{z\left(D_q^{\delta} H(z)\right)''}{\left(D_q^{\delta} H(z)\right)'} \right) - \gamma cos\lambda \right\} \right| d\theta \leq k\pi cos\lambda \sum_{j=1}^{m} \alpha_j (1 - \xi_j)$$

Subsequently, $H \in V_k^{\lambda}(\gamma, b, \delta, q)$ with γ is given by (2.1).

By setting $\gamma = 0$, $\delta = q - 1$ in Theorem 2.2, we obtain the following result.

Corollary 2.2 Let $f_j \in V_k(\xi_j, b)\ for\ j \in \{1, 2, \dots, m\}$ with $(0 \leq \xi_j \leq 1), b \in \mathbb{C} - \{0\}$. Also let $\alpha_j > 0$ $(j \in \{1, 2, \dots, m\})$. If

$$0 \leq 1 + \sum_{j=1}^{m} \alpha_j (\xi_j - 1) < 1,$$

then the integral operator H defined (1.8) is in the class $V_k\ (\gamma, b)$ *with*

$$\gamma = 1 + \sum_{j=1}^{m} \alpha_j (\xi_j - 1)$$

Remark 2.1 In Corollary 2.2, setting

i) $\xi_1 = \xi_2 = \cdots \xi_m = \xi$, we have [[10], Theorem 2.5].

ii) *k=2, we have [[21], Theorem 3].*

Remark 2.2 *other work related to q-hypergeometric function and analytic functions can be found in [22], [23], [24], [25], [26]*

References

[1] L. Spacek, Prispêvek k teorii funkei prostych, Ĉapopis pro pê_stovani matematiky a fysiky, vol. 62, pp. 1219, 1933.
[2] M.S. Robertson, Univalent functions f(z) for which $zf'(z)$ is spirallike,The Michigan Mathematical Journal, vol. 16, pp. 97101, 1969.
[3] K. S. Padmanabhan and R. Parvatham, Properties of a class of functions with bounded boundary rotation, Annales Polonici Mathematici, vol. 31, no. 3, pp. 311323, 1975.
[4] B. Pinchuk, Functions of bounded boundary rotation, Israel Journal of Mathematics, vol. 10, pp. 716, 1971.
[5] G. Gasper and M. Rahman, Basic Hypergeometric Series, vol. 35 of Encylopedia of Mathematics and its Applications, Cambridge UniversityPress,Cambridge , Uk, 1990.
[6] H. Exton, q- Hypergeometric Functions and Applications, Ellis Horwood, Chichester , UK, 1983.
[7] S. Ruscheweyh, New criteria for univalent functions, Proceedings of the American Mathematical Society, vol. 49, pp. 109115, 1975.
[8] K. I. Noor, M. Arif, and A. Muhammad, Mapping properties of some classes of analytic functions under an integral operator, Journal of Mathematical Inqualities, vol. 4, no. 4, pp. 593600, 2010.
[9] E. J. Moulis Jr., Generalizations of the Robertson function, Pacific Journal of Mathematics, vol, 81. no. 1, pp. 167174, 1979.
[10]K.I Noor, M. Arif, and W. U. Haq, Some properties of certain integral operators, Acta Universitatis Apulensis. Mathematics. Informatics, no. 21, pp. 8995, 2010.
[11] B. A. Frasin, Family of analytic functions of complex order, Acta Mathematica Academiae Paedagogiace Nyi'regyha' ziensis (N.S), vol. 22, no. 2, pp. 179191, 2006.
[12] O. Tammi, On the maximization of the cofficients of schlicht and related functions, Ann. Acad. Sci. Fenn. Ser. AI, Math. Phys., 114 (1952), 51 pages.
[13]V. Paatero, Uber gebiete von beschrankter randdehung, Ann. Acad. Sci.Fenn. Ser. A. , 37 (1933), 20 pages.
[14] Georgia Irina Oros, Gheorghe Oros, and Daniel Breaz, Sufficient conditions for univalence of an integral operator, Journal of inequalities and applications, vol. 2008, Article ID 127645, 7 pages.
[15] D. Breaz and N. Breaz, Two integral operators, Stusia Universitatis Babes- Bolyai Mathematica, vol. 47, no. 3, pp. 1319, 2002.

[16]S.S Miller, P. T. Mocanu, and M. O. Reade, Starlike integral operators, Pacific Journal ofMathematics, vol. 79, no.1, pp. 157168, 1978.
[17] I. W. Alexander, Functions which map the interior of the unite circle upon simple regions, Annals of Mathematics, vol. 17, no.1, pp. 1222, 1915.
[18] D. Breaz, S. Owa, and N. Breaz, A new integral univalent operator, Acta Universitatis Apulensis. Mathematics. Information, no.16, pp. 1116, 2008.
[19] J.A. Pfaltzgraff, Univalence of the integral of $f'(z)^{\lambda}$, The Bulletin of the London Mathematical Society, vol. 7, no.3, pp.254256, 1975.
[20] N. N. Pascu and V. Pescar, On the integral operators of Kim- Merkes and Pfaltzgraff, Mathematica (Cluj), vol. 32(55), no.2, pp. 185192, 1990.
[21] S. Bulut, A note on the paper of Breaz and Gney, Journal of Mathematical Inequalities, vol. 2, no.4, pp. 549553, 2008.
[22] A Mohammed and M. Darus, A generalized operator involving the q- hypergeometric function, *Mathematici Vesnik*, A valiable online 10.06.2012. 12 pages.
[23] H. Aldweby and M. Darus, Univalence of a New General Integral Operator Associated with the q- Hypergeometric Function, *International Journal of Mathematics and Mathematical Sciences*, vol. 2013, Article ID 769537, 5 pages.
[24] H. Aldweby and M. Darus,ASubclass of Harmonic Univalent Functions Associated with the q- Analogue of Dziok- Srivatava Operator, ISRN Mathematical Analysis, vol. 2013, Article ID 382312, 6 pages.
[25] H. Aldweby and M. Darus ,A Subclass of Harmonic Meromorphic Functions Associated with Basic Hypergeometric Function, The Scientific World Journal, vol. 2013, Article ID 164287, 7 pages.
[26] H. Aldweby and M. Darus, "Some Subordination Result on q- Analogue of Ruscheweyh Differential Operator" , *Abstrct and AppliedAnalysis*, *vol 2014*, Article ID 958563, 6 pages.

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