مجلة التربوي
Journal of Educational
ISSN: 2011- 421X
Arcif Q3

معامل التأثير العربي 1.5
العدد 20

مجلة التربوي

مجلة علمية محكمة تصدر عن كلية التربية

جامعة المرقب

العدد العشرون

يناير 2022م

هيئـــة تحريـر

مجلة التربوي

- المجلة ترحب بما يرد عليها من أبحاث وعلى استعداد لنشرها بعد التحكيم .
- المجلة تحترم كل الاحترام آراء المحكمين وتعمل بمقتضاها .
- كافة الآراء والأفكار المنشورة تعبر عن آراء أصحابها ولا تتحمل المجلة تبعاتها .
- يتحمل الباحث مسؤولية الأمانة العلمية وهو المسؤول عما ينشر له .
- البحوث المقدمة للنشر لا ترد لأصحابها نشرت أو لم تنشر .

مجـلة الـتربـوي
Journal of Educational
ISSN: 2011- 421X
Arcif Q3

معامل التأثير العربي 1.5
العدد 20

ضوابط النشر:

يشترط في البحوث العلمية المقدمة للنشر أن يراعى فيها ما يأتي :

- أصول البحث العلمي وقواعده .
- ألا تكون المادة العلمية قد سبق نشرها أو كانت جزءا من رسالة علمية .
- يرفق بالبحث تزكية لغوية وفق أنموذج معد .
- تعدل البحوث المقبولة وتصحح وفق ما يراه المحكمون .
- التزام الباحث بالضوابط التي وضعتها المجلة من عدد الصفحات ، ونوع الخط ورقمه ، والفترات الزمنية الممنوحة للتعديل ، وما يستجد من ضوابط تضعها المجلة مستقبلا .

تنبيهات :

- للمجلة الحق في تعديل البحث أو طلب تعديله أو رفضه .
- يخضع البحث في النشر لأولويات المجلة وسياستها .
- البحوث المنشورة تعبر عن وجهة نظر أصحابها ، ولا تعبر عن وجهة نظر المجلة .

Information for authors

1- Authors of the articles being accepted are required to respect the regulations and the rules of the scientific research.
2- The research articles or manuscripts should be original and have not been published previously. Materials that are currently being considered by another journal or is a part of scientific dissertation are requested not to be submitted.
3- The research articles should be approved by a linguistic reviewer.
4- All research articles in the journal undergo rigorous peer review based on initial editor screening.
5- All authors are requested to follow the regulations of publication in the template paper prepared by the editorial board of the journal.

Attention
1- The editor reserves the right to make any necessary changes in the papers, or request the author to do so, or reject the paper submitted.
2- The research articles undergo to the policy of the editorial board regarding the priority of publication.
3- The published articles represent only the authors' viewpoints.

On the fine spectrum of the generalized difference operator $B(r,s)$ over the Hahn sequence space h

Suad H. Abu-Janah
Faculty of Science, Elmergib University, Msallata
suadabujanah@yahoo.com

Abstract

Of particular interest to many researchers are the investigation of the spectra of the difference operator and its generalizations over sequence spaces. In this work, we will introduce the spectra analysis of the generalized difference operator $B(r,s)$ on the Hahn sequence space h. Moreover, we will improve some proofs of the results in the existing literature [27].
Keywords: Spectrum, Infinite matrices, Sequence spaces.

1 Introduction and preliminaries

The generalized difference operator $B(r,s): \mu \to \mu$ is defined on the Banach sequence space μ as:

$$B(r,s)x := (rx_0, rx_1 + sx_0, rx_2 + sx_1, \dots),\ x = (x_k)_{k=0}^{\infty} \in \mu, \quad \textbf{(1.1)}$$

where $r, s \in \mathbb{R}$, $s \neq 0$.

This operator can be represented by a band matrix as

$$B(r,s) = \begin{bmatrix} r & 0 & 0 & \cdots \\ s & r & 0 & \cdots \\ 0 & s & r & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (s \neq 0).$$

In fact, the operator $B(r,s)$ is reduced to the right-shift, difference and Zweier matrices in the special cases $(r,s) = (0,1)$, $(r,s) = (1,-1)$ and $(r,s) = (r, 1-r)$, respectively.

The fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces c_0 and c have been studied by Altay and Başar [4]. Also, the spectrum and fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces ℓ_1 and bv have been studied by Furkan et al. [10]. Recently, the fine spectrum of the operator B(r,s) over ℓ_p and bv_p where $1 < p < \infty$ has been examined by Bilgiç and

Furkan [7]. Dutta and Tripathy [8] have studied the fine spectrum of the generalized difference operator $B(r,s)$ over the space of convergent series. Recently, the fine spectrum of the operator $B(r,s)$ over bv_0 has been studied by El-Shabrawy and Abu-Janah [9]. Moreover, some additional results concerning on other well-known classification of the spectrum of $B(r,s)$ have been given as the approximate point spectrum, defect spectrum and compression spectrum, see [6].

Indeed, if $r=1$ and $s=-1$, the operator $B(r,s)$ reduces to the operator Δ. The fine spectrum of the difference operator Δ over the sequence spaces c_0 and c has been studied by Altay and Başar [3] and over ℓ_1 and bv by Kayaduman and Furkan [15]. Recently, the fine spectra of the difference operator Δ over the sequence spaces ℓ_p and bv_p, where $1 \le p < \infty$, has been studied by Akhmedov and Başar [1,2]. In 2016, the fine spectrum of the forward difference operator on the Hahn space h was studied by Yeşilkayagil and Kirişci [27].

Now, we introduce some definitions and notations.

Let X be a complex infinite dimensional Banach space and $B(X)$ be the set of all bounded linear operators on X into itself. If $T \in B(X)$, we use $R(T)$ to denote the range of T.

For a Banach space X we use X^* to denote the dual space of X. If $T \in B(X)$, then $T^* \in B(X^*)$ is the adjoint operator of T. With T we associate the operator

$$T_\lambda = T - \lambda I$$

where I is the identity mapping of X onto itself. If T_λ has an inverse which is linear, we denote it by T_λ^{-1} and call it the resolvent operator of T.

If $T \in B(X)$, All of the points λ in the complex plane $\mathbb{C}$ are divided into two mutually exclusive and complementary sets:

The resolvent set. $\rho(T,X) = \{\lambda \in \mathbb{C}: T - \lambda I \text{ is a bijection}\}$,

and

The spectrum: $\sigma(T,X) = \{\lambda \in \mathbb{C}: T - \lambda I \text{ is not invertible}\}$,

The spectrum: $\sigma(T,X)$ is the complement of $\rho(T,X)$ in the complex plane $\mathbb{C}$.

It is useful to make a finer classification of points by subdividing $\sigma(T,X)$ in some way. One such method of subdivision is well-known, the spectrum $\sigma(T,X)$ can be analyzed into three disjoint sets as follows:

The point spectrum: $\sigma_p(T,X) = \{\lambda \in \mathbb{C}: T - \lambda I \text{ is not injective}\}$,

The continuous spectrum:
$\sigma_c(T,X) = \{\lambda \in \mathbb{C}: T-\lambda I \text{ is injective and } \overline{R(T-\lambda I)} = X, \text{but } R(T-\lambda I) \neq X\}$,
The residual spectrum:
$\sigma_r(T,X) = \{\lambda \in \mathbb{C}: T-\lambda I \text{ is injective, but } \overline{R(T-\lambda I)} \neq X\}$,
These three subspectra form a disjoint subdivisions

$$\sigma(T,X) = \sigma_p(T,X) \cup \sigma_c(T,X) \cup \sigma_r(T,X).$$

This subdivision is customary subdivision (see, for example, Stone [23] or [18,19]). An advantage of this classification is the division of the spectrum into disjoint sets.

Following Taylor and Halberg [24,25], a linear operator T with domain and range in a normed space X, is calssified I, II or III, according as its range, $R(T)$, is all of X; is not all of X, but is dense in X; or is not dense in X. In addition T is classified 1, 2, or 3 according as T^{-1} exists and is continuous; exists, but is not continuous; or does not exist. The state of an operator is the combination of its Roman and Arabic numerical classification and is denoted by the Roman numeral with the Arabic numeral as a subscript [24,p.94], [25,p.235-236].

For a bounded linear operator T on a complex Banach space X we partition the complex plane into subsets corresponding to the states of the operator $T-\lambda I$. For example, the subset consisting of those λ for which the state of the operator $T-\lambda I$ is II_3 will be denoted by $II_3(T,X)$. Thus the resolvent set, $\rho(T,X)$, of the operator T consists of the union of $I_1(T,X)$ and $II_1(T,X)$, the point spectrum consists of the union of $I_3(T,X)$, $II_3(T,X)$ and $III_3(T,X)$, the residual spectrum consists of the union of $III_1(T,X)$ and $III_2(T,X)$ and the continuous spectrum consists of the $II_2(T,X)$[24,p.109],[25,p.264-265].

Following Appel et al. [5], three more subdivisions of the spectrum can be defined, which are not necessarily disjoint: the approximate point spectrum, defect spectrum and compression spectrum.

Definition 1.1: Given a bounded linear operator T on a Banach space X, we call a sequence (x_k) in X as a *Weyl sequence* for T if $\|x_k\| = 1$ and $\|Tx_k\| \to 0$, as $\kappa \to \infty$.

Definition 1.2: In what follows, we call the set

$$\sigma_{\text{ap}}(T,X) = \{\lambda \epsilon \mathbb{C}: \text{there exists a Weyl sequence for } T-\lambda I\}$$

the approximate point spectrum of T. Moreover, the subspectrum

$$\sigma_\delta(T,X)=\{\lambda\in\mathbb{C}:\mathcal{R}(\lambda I-T)\neq X\}$$

is called defect spectrum of T.

There is another subspectrum

$$\sigma_{co}(T,X)=\{\lambda\in\mathbb{C}:\overline{\mathcal{R}(\lambda I-T)}\neq X\},$$

which is often called compression spectrum.

The two subspectra $\sigma_{\text{ap}}(T,X)$ and $\sigma_\delta(T,X)$ are not necessarily disjoint. As well as $\sigma_{\text{ap}}(T,X)$ and $\sigma_{co}(T,X)$ are not necessarily disjoint.

Where

$$\sigma(T,X)=\sigma_{\text{ap}}(T,X)\cup\sigma_\delta(T,X)$$
$$\sigma(T,X)=\sigma_{\text{ap}}(T,X)\cup\sigma_{co}(T,X)$$

Clearly, $\sigma_p(T,X)\subseteq\sigma_{\text{ap}}(T,X)$ and $\sigma_{co}(T,X)\subseteq\sigma_\delta(T,X)$. Moreover, comparing these subspectra with those in $\sigma(T,X)=\sigma_p(T,X)\cup\sigma_c(T,X)\cup\sigma_r(T,X)$ we note that

$$\sigma_r(T,X)=\sigma_{co}(T,X)\setminus\sigma_p(T,X),$$
$$\sigma_r(T,X)=\sigma(T,X)\setminus\left[\sigma_p(T,X)\cup\sigma_{co}(T,X)\right].$$

The following result clarifies the connection between the spectrum of a bounded linear operator and that of its adjoint which is needed in the sequel.

Proposition 1.1. [5] Spectra and subspectra of an operator $T\in B(X)$ and its adjoint $T^*\in B(X^*)$ are related by the following relations :

(a) $\sigma(T^*,X^*)=\sigma(T,X)$,

(b) $\sigma_c(T^*,X^*)\subseteq\sigma_{\text{ap}}(T,X)$,

(c) $\sigma_{\text{ap}}(T^*,X^*)=\sigma_\delta(T,X)$,

(d) $\sigma_\delta(T^*,X^*)=\sigma_{\text{ap}}(T,X)$,

(e) $\sigma_p(T^*,X^*)=\sigma_{co}(T,X)$,

(f) $\sigma_{co}(T^*,X^*)\supseteq\sigma_p(T,X)$,

(g) $\sigma(T,X)=\sigma_{\text{ap}}(T,X)\cup\sigma_p(T^*,X^*)=\sigma_p(T,X)\cup\sigma_{\text{ap}}(T^*,X^*)$.

The following relations can be obtained by the preceding definitions

$\sigma_{\text{ap}}(T,X) = \sigma(T,X)\backslash III_1(T,X),$

$$\sigma_\delta(T,X) = \sigma(T,X)\backslash I_3(T,X). \qquad \textbf{(1.2)}$$

In [12,13] the Hahn space h is defined by

$$h = \{x = (x_k)_{k=0}^{\infty}: \lim_{k\to} x_k = 0 \text{ and } \sum_{k=0}^{\infty} k|x_{k+1} - x_k| < \infty\},$$

with the norm

$$\|x\|_h = \sum_{k=0}^{\infty} k|x_{k+1} - x_k| + \sup_k |x_k|.$$

Rao [20 ,Proposition 2.1] defined a new norm on h as $\|x\|_h = \sum_{k=0}^{\infty} k|x_{k+1} - x_k|$. Further, the dual space of h is norm isomorphic to the space σ_∞ of all absolutely summable sequences $x = (x_k)_{k=0}^{\infty}$, which is defined as

$$\sigma_\infty = \{x = (x_k)_{k=0}^{\infty}: \sup_{n\in\mathbb{N}} (1/n + 1)\,|\sum_{k=0}^{n} x_k\,| < \infty\}.$$

The spaces h and σ_∞ are Banach spaces with the given norm in [13,20].

Theorem 1.1. [13,16,17**:** The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(h)$ if and only if

1. $\lim_{n\to\infty} a_{nk} = 0$, for all $k = 1,2,\ldots,$
2. $\sum_{n=1}^{\infty} n\left|a_{nk} - a_{n+1,k}\right|$converges, for all $k = (1,2,\ldots)$.
3. $\sup_k \frac{1}{k} \sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k} (a_{nv} - a_{n+1,v})\right| < \infty.$

Main Results

Our study in this section is focused on the fine spectrum of B(r,s) on the Hahn space h, which is defined by Eq. (1.1), where µ=h.

The following theorem shows the bounded linearity of the operator B(r,s) on h.

Theorem 2.1. The operator $B(r,s): h \to h$ is a bounded linear operator.

Proof. The linearity of $B(r,s)$ is trivial and so is omitted here for brevity. To show the operator $B(r,s)$ is a bounded linear transformation on h into itself, it is enough to prove that $B(r,s)$ satisfies the three conditions given by Theorem 1.1.

Obviously, the matrix $B(r,s) = (b_{nk})$ satisfies

. $\lim_{n\to\infty} b_{nk} = 0, k = 1,2,3, \ldots.$

Also, let

$$R_k = \sum_{n=1}^{\infty} n\left|b_{nk} - b_{n+1,k}\right|, \quad k = 1,2,3, \ldots.$$

So

$R_1 = |b_{11} - b_{21}| + 2|b_{21} - b_{31}| + 3|b_{31} - b_{41}| + 4|b_{41} - b_{51}| + \cdots = |r - s| + 2|s|,$ $R_2 = |b_{12} - b_{22}| + 2|b_{22} - b_{32}| + 3|b_{32} - b_{42}| + 4|b_{42} - b_{52}| + \cdots$
= |r|+2|r-s|+3|s|,

and

$$R_3 = |b_{13} - b_{23}| + 2|b_{23} - b_{33}| + 3|b_{33} - b_{43}| + 4|b_{43} - b_{53}| + \cdots = 2|r| + 3|r - s| + 4|s|.$$

Then, in general, we obtain

$$R_k = \sum_{n=1}^{\infty} n\left|b_{nk} - b_{n+1,k}\right| = (k-1)|r| + k|r - s| + (k+1)|s|,$$

which is convergent, for each fixed $k \in \mathbb{N}$. Additionally, let

$$S_k = \sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k}(b_{nv} - b_{n+1,v})\right|, \quad k = 1,2,3, \cdots.$$

So

$$S_1 = \sum_{n=1}^{\infty} n\left|b_{n1} - b_{n+1,1}\right| = |b_{11} - b_{21}| + 2|b_{21} - b_{31}| + 3|b_{31} - b_{41}| + 4|b_{41} - b_{51}| + \cdots = |r - s| + 2|s|,$$

$$S_2 = \sum_{n=1}^{\infty} n\left|(b_{n1} - b_{n+1,1}) + (b_{n2} - b_{n+1,2})\right| = |b_{11} - b_{21} + b_{12} - b_{22}| + 2|b_{21} - b_{31} + b_{22} - b_{32}| + 3|b_{31} - b_{41} + b_{32} - b_{42}| + 4|b_{41} - b_{51} + b_{42} - b_{52}| + \cdots$$

$= 2|r| + 4|s|,$

and

$$S_3 = \sum_{n=1}^{\infty} n\left|(b_{n1} - b_{n+1,1}) + (b_{n2} - b_{n+1,2}) + (b_{n3} - b_{n+1,3})\right|$$

$= |b_{11} - b_{21} + b_{12} - b_{22} + b_{13} - b_{23}| + 2|b_{21} - b_{31} + b_{22} - b_{32} + b_{23} - b_{33}| +3|b_{31} - b_{41} + b_{32} - b_{42} + b_{33} - b_{43}| + 4|b_{41} - b_{51} + b_{42} - b_{52} + b_{43} - b_{53}| +5|b_{51} - b_{61} + b_{52} - b_{62} + b_{53} - b_{63}| + \cdots$

$= 3|r|+5|s|.$

Precisely, we have the following cases:

For $n = 1$,

$$\left|\sum_{v=1}^{k}(b_{1v} - b_{2v})\right| = |b_{11} - b_{21} + b_{12} - b_{22} + b_{13} - b_{23} + \cdots + b_{1k} - b_{2k}|$$
$$= |(b_{11} + b_{12} + b_{13} + \cdots + b_{1k}) - (b_{21} + b_{22} + b_{23} + \cdots + b_{2k})|$$
$$= |r - (s + r)|$$
$$= |s|.$$

For $n = k > 1$,

$$\left|\sum_{v=1}^{k}(b_{kv} - b_{k+1,v})\right| = |b_{k1} - b_{k+1,1} + b_{k2} - b_{k+1,2} + \cdots + b_{kk} - b_{k+1,k}|$$
$$= \left|(b_{k1} + b_{k2} + \cdots + b_{kk}) - (b_{k+1,1} + b_{k+1,2} + \cdots + b_{k+1,k})\right|$$
$$= |(s + r) - s|$$
$$= |r|.$$

For $n = k + 1 > 1$,

$$\left|\sum_{v=1}^{k}(b_{k+1,v} - b_{k+2,v})\right| = |b_{k+1,1} - b_{k+2,1} + b_{k+1,2} - b_{k+2,2} + \cdots + b_{k+1,k} - b_{k+2,k}|$$

$$= \left|(b_{k+1,1} + b_{k+1,2} + \cdots + b_{k+1,k}) - (b_{k+2,1} + b_{k+2,2} + \cdots + b_{k+2,k})\right|$$
$$= |s-0|$$
$$= |s|.$$

But, when $n \neq 1, k, k+1$, we have

$$\left|\sum_{v=1}^{k}(b_{nv} - b_{n+1,v})\right| = \left|(b_{n1} + b_{n2} + \cdots + b_{nk}) - (b_{n+1,1} + b_{n+1,2} + \cdots + b_{n+1,k})\right|$$
$$= |0 - 0| = 0.$$

Therefore

$$\left|\sum_{v=1}^{k}\left(b_{nv} - b_{n+1,v}\right)\right| = |s| \quad \text{when } n = 1 \text{ } or \text{ } n = k+1$$
$$\left|\sum_{v=1}^{k}\left(b_{nv} - b_{n+1,v}\right)\right| = |r| \quad \text{when } n = k$$
$$\left|\sum_{v=1}^{k}\left(b_{nv} - b_{n+1,v}\right)\right| = 0 \quad \text{otherwise}$$

Then, for $k \geq 2$

$$S_k = \sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k}(b_{nv} - b_{n+1,v})\right| = |s| + k|r| + (k+1)|s|$$
$$= k|r| + (k+2)|s|.$$

Thus

$$\sup_{k\geq 2} \frac{1}{k}\sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k}(b_{nv} - b_{n+1,v})\right|$$

$= \sup_{k\geq 2}[|r| + ((k+2)/k)|s|] = |r| + 2|s| < \infty.$
So, the operator $B(r,s)$ is bounded.
This completes the proof. ■

Theorem 2.2. $\sigma(B(r,s),h) = \{\lambda \in \mathbb{C}: |\lambda - r| \leq |s|\}.$
Proof. Suppose $\lambda \notin \{\lambda \in \mathbb{C}: |\lambda - r| \leq |s|\}$. Then $|\lambda - r| > |s|$, and so, the matrix $B(r,s) - \lambda I$ has an inverse.
If $y = (y_k)_{k=1}^{\infty} \in h$; solving the eqution $(B(r,s) - \lambda I)x = y$, for $x = (x_k)_{k=1}^{\infty}$ in terms of y, we get

$$x_1 = (1/(r-\lambda))y_1$$

$$x_1 = \frac{1}{r-\lambda} y_1$$

$$x_k = \frac{(-s)^{k-1}}{(r-\lambda)^k} y_1 + \frac{(-s)^{k-2}}{(r-\lambda)^{k-1}} y_2 + \cdots + \frac{-s}{(r-\lambda)^2} y_{k-1} \frac{1}{(r-\lambda)} y_k, \quad k \geq 2$$

مجـلة الـتربـوي
Journal of Educational
ISSN: 2011- 421X
Arcif Q3
معامل التأثير العربي 1.5
العدد 20

Then $(B(r,s)-\lambda I)^{-1}=(c_{nk})$, where

$$(c_{nk})=\begin{bmatrix} \frac{1}{r-\lambda} & 0 & 0 & 0 & 0 & \cdots \\ \frac{-s}{(r-\lambda)^2} & \frac{1}{r-\lambda} & 0 & 0 & 0 & \cdots \\ \frac{s^2}{(r-\lambda)^3} & \frac{s^2}{(r-\lambda)^3} & \frac{1}{r-\lambda} & 0 & 0 & \cdots \\ \frac{-s^3}{(r-\lambda)^4} & \frac{s^2}{(r-\lambda)^3} & \frac{-s}{(r-\lambda)^2} & \frac{1}{r-\lambda} & 0 & \cdots \\ \frac{s^4}{(r-\lambda)^5} & \frac{-s^3}{(r-\lambda)^4} & \frac{s^2}{(r-\lambda)^3} & \frac{-s}{(r-\lambda)^2} & \frac{1}{r-\lambda} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad (2.1)$$

Hence

$$\lim_{n\to\infty}\left|\frac{c_{n+1,k}}{c_{nk}}\right|=\left|\frac{s}{r-\lambda}\right|<1,$$

And so

$$\lim_{n\to\infty} c_{nk}=0,\quad k=1,2,3,\dots\dots$$

Let

$$R_k=\sum_{n=1}^{\infty} n|c_{nk}-c_n+1,k|$$

Then, the details are given as follows:

$$R_1=\sum_{n=1}^{\infty} n|c_{n1}-c_{n+1,1}|$$
$$=|c_{11}-c_{21}|+2|c_{21}-c_{31}|+3|c_{31}-c_{41}|+$$
$$4|c_{41}-c_{51}|+\cdots\dots\dots\dots$$
$$=\left|\frac{1}{r-\lambda}\right|\left|1-\frac{s}{r-\lambda}\right|+2\left|\frac{s}{(r-\lambda)^2}\right|\left|1-\frac{s}{r-\lambda}\right|$$
$$+3\left|\frac{s^2}{(r-\lambda)^3}\right|\left|1-\frac{s}{r-\lambda}\right|+\cdots$$
$$=\left|\frac{1}{r-\lambda}\right|\left|1-\frac{s}{r-\lambda}\right|$$
$$+\left(1+2\left|\frac{s}{r-\lambda}\right|+3\left|\frac{s}{r-\lambda}\right|^2+\cdots\dots.\right)$$

$$R_2=\sum_{n=1}^{\infty} n|c_{n2}-c_{n+1,2}|$$
$$=|c_{12}-c_{22}|+2|c_{22}-c_{32}|+3|c_{32}-c_{42}|+$$
$$4|c_{42}-c_{52}|+\cdots$$

$$= \left|\frac{1}{r-\lambda}\right| 2 + \left|\frac{1}{r-\lambda}\right| \left|1 - \frac{s}{r-\lambda}\right|$$

$$+ 3\left|\frac{s}{(r-\lambda)^2}\right| \left|1 - \frac{s}{r-\lambda}\right| + 4\left|\frac{s^2}{(r-\lambda)^3}\right| \left|1 - \frac{s}{r-\lambda}\right| + \cdots$$

$$= \left|\frac{1}{r-\lambda}\right| + \left|\frac{1}{r-\lambda}\right| \left|1 - \frac{s}{r-\lambda}\right|$$

$$+ \left(2 + 3\left|\frac{s}{r-\lambda}\right| + 4\left|\frac{s}{r-\lambda}\right|^2 + \cdots \ldots.\right)$$

$$R_3 = \sum_{n=1}^{\infty} n|c_{n3} - c_{n+1,3}|$$

$$= |c_{13} - c_{23}| + 2|c_{23} - c_{33}| + 3|c_{33} - c_{43}| + 4|c_{43} - c_{53}| + \cdots$$

$$= 2\left|\frac{1}{r-\lambda}\right| + 3\left|\frac{1}{r-\lambda}\right| \left|1 - \frac{s}{r-\lambda}\right|$$

$$+ 4\left|\frac{s}{(r-\lambda)^2}\right| \left|1 - \frac{s}{r-\lambda}\right| + 5\left|\frac{s^2}{(r-\lambda)^3}\right| \left|1 - \frac{s}{r-\lambda}\right| + \cdots$$

$$= 2\left|\frac{1}{r-\lambda}\right| + \left|\frac{1}{r-\lambda}\right| \left|1 - \frac{s}{r-\lambda}\right|$$

$$+ \left(3 + 4\left|\frac{s}{r-\lambda}\right| + 5\left|\frac{s}{r-\lambda}\right|^2 + \cdots \ldots.\right)$$

Then, one can obtain that

$$R_k = (k-1)\left|\frac{1}{r-\lambda}\right| + \left|\frac{1}{r-\lambda}\right| \left|1 - \frac{s}{r-\lambda}\right| \left(\sum_{n=0}^{\infty} (k+n)\left|\frac{s}{r-\lambda}\right|^n\right)$$

The ratio test yields that the series $\sum_{n=0}^{\infty}(k+n)\left|\frac{s}{r-\lambda}\right|^n$ is convergent for fixed
$k = 1,2,3, \ldots$

It remains now to prove that

$$\sup_k \frac{1}{k} \sum_{n=1}^{\infty} n \left|\sum_{v=1}^{k} (c_{nv} - c_{n+1,v})\right| < \infty$$

For this purpose let

$$S_k = \sum_{n=1}^{\infty} n \left|\sum_{v=1}^{k} (c_{nv} - c_{n+1,v})\right|, \quad k = 1,2,3, \ldots$$

Then

$$S_1 = \sum_{n=1}^{\infty} n|c_{n1} - c_{n+1,1}|$$

$$= |c_{11} - c_{21}| + 2|c_{21} - c_{31}| + 3|c_{31} - c_{41}| + 4|c_{41} - c_{51}| + \cdots \ldots \ldots \ldots.$$

$$= \left|\frac{1}{r-\lambda} - \frac{s}{(r-\lambda)^2}\right| + 2\left|\frac{s}{(r-\lambda)^2} - \frac{s^2}{(r-\lambda)^3}\right| + 3\left|\frac{s^2}{(r-\lambda)^3} - \frac{s^3}{(r-\lambda)^4}\right| + \cdots$$

$$= \left|\frac{1}{r-\lambda}\right|\left|1 - \frac{s}{r-\lambda}\right| + \left(1 + 2\left|\frac{s}{r-\lambda}\right| + 3\left|\frac{s}{r-\lambda}\right|^2 + \cdots \ldots .\right)$$

$$= \left|\frac{1}{r-\lambda}\right|\left|1 - \frac{s}{r-\lambda}\right| + \left(\sum_{n=0}^{\infty}(n+1)\left|\frac{s}{r-\lambda}\right|^n\right),$$

$$S_2 = \sum_{n=1}^{\infty} n\left|c_{n1} - c_{n+1,1+} + c_{n2} - c_{n+1,2}\right|$$

$$= |c_{11} - c_{21} + c_{12} - c_{22}| + 2|c_{21} - c_{31} + c_{22} - c_{32}| + 3|c_{31} - c_{41} + c_{32} - c_{42}| + 4|c_{41} - c_{51} + c_{42} - c_{52}| + \cdots \ldots \ldots \ldots.$$

$$= \left|\frac{1}{r-\lambda} - \frac{s}{(r-\lambda)^2} + 0 - \frac{1}{r-\lambda}\right| + 2\left|\frac{s}{(r-\lambda)^2} - \frac{s^2}{(r-\lambda)^3} + \frac{1}{r-\lambda} - \frac{s}{(r-\lambda)^2}\right| + 3\left|\frac{s^2}{(r-\lambda)^3} - \frac{s^3}{(r-\lambda)^4} + \frac{s}{(r-\lambda)^2} - \frac{s^2}{(r-\lambda)^3}\right| + 4\left|\frac{s^3}{(r-\lambda)^4} - \frac{s^4}{(r-\lambda)^5} + \frac{s^2}{(r-\lambda)^3} - \frac{s^3}{(r-\lambda)^4}\right| + \cdots$$

$$= \left|\frac{s}{(r-\lambda)^2}\right| + 2\left|\frac{s}{r-\lambda} - \frac{s^2}{(r-\lambda)^3}\right| + 3\left|\frac{s}{(r-\lambda)^2} - \frac{s^3}{(r-\lambda)^4}\right| + 4\left|\frac{s^2}{(r-\lambda)^3} - \frac{s^4}{(r-\lambda)^5}\right| + \cdots$$

$$= \left|\frac{s}{(r-\lambda)^2}\right| + \left|\frac{1}{r-\lambda}\right|\left|1 - \frac{s^2}{(r-\lambda)^2}\right|\left(2 + 3\left|\frac{s}{r-\lambda}\right| + 4\left|\frac{s}{r-\lambda}\right|^2 + \cdots\right)$$

مجـلة الـتربـوي
Journal of Educational
ISSN: 2011- 421X
Arcif Q3

معامل التأثير العربي 1.5
العدد 20

$$= \left|\frac{s}{(r-\lambda)^2}\right| + \left|\frac{1}{r-\lambda}\right| \left|1 - \frac{s^2}{(r-\lambda)^2}\right| + \left(\sum_{n=0}^{\infty}(n+2)\left|\frac{s}{r-\lambda}\right|^n\right),$$

And

$$S_3 = \sum_{n=1}^{\infty} n|c_{n1} - c_{n+1,1+} + c_{n2} - c_{n+1,2} + c_{n3} - c_{n+1,3}|$$
$$= |c_{11} - c_{21} + c_{12} - c_{22} + +c_{13} - c_{23}| + 2|c_{21} - c_{31} + c_{22} - c_{32} + c_{23} - c_{33}| + 3|c_{31} - c_{41} + c_{32} - c_{42} + c_{33} - c_{43}| + 4|c_{41} - c_{51} + c_{42} - c_{52} + c_{43} - c_{53}| + \cdots \ldots \ldots ....$$

$$= \left|\frac{1}{r-\lambda} - \frac{s}{(r-\lambda)^2} + 0 - \frac{1}{r-\lambda} + 0 - 0\right|$$
$$+2\left|\frac{s}{(r-\lambda)^2} - \frac{s^2}{(r-\lambda)^3} + \frac{1}{r-\lambda} - \frac{s}{(r-\lambda)^2} + 0 - \frac{1}{r-\lambda}\right|$$
$$+3\left|\frac{s^2}{(r-\lambda)^3} - \frac{s^3}{(r-\lambda)^4} + \frac{s}{(r-\lambda)^2} - \frac{s^2}{(r-\lambda)^3} + \frac{1}{r-\lambda} - \frac{s}{(r-\lambda)^2}\right|$$
$$+4\left|\frac{s^3}{(r-\lambda)^4} - \frac{s^4}{(r-\lambda)^5} + \frac{s^2}{(r-\lambda)^3} - \frac{s^3}{(r-\lambda)^4} + \frac{s}{(r-\lambda)^2} - \frac{s^2}{(r-\lambda)^3}\right| + \cdots$$

$$= \left|\frac{s}{(r-\lambda)^2}\right| + 2\left|\frac{s^2}{(r-\lambda)^3}\right| + 3\left|\frac{1}{r-\lambda} - \frac{s^3}{(r-\lambda)^4}\right| + 4\left|\frac{s}{(r-\lambda)^2} - \frac{s^4}{(r-\lambda)^5}\right| + \cdots$$

$$= \left|\frac{s}{(r-\lambda)^2}\right| + 2\left|\frac{s^2}{(r-\lambda)^3}\right| + \left|\frac{1}{r-\lambda}\right| \left|1 - \frac{s^3}{(r-\lambda)^3}\right| \left(3 + 4\left|\frac{s}{r-\lambda}\right| + 5\left|\frac{s}{r-\lambda}\right|^2 + \cdots\right)$$

$$= \left|\frac{s}{(r-\lambda)^2}\right| + 2\left|\frac{s^2}{(r-\lambda)^3}\right| + \left|\frac{1}{r-\lambda}\right| \left|1 - \frac{s^3}{(r-\lambda)^3}\right| + \left(\sum_{n=0}^{\infty}(n+3)\left|\frac{s}{r-\lambda}\right|^n\right),$$

Recursively, we obtain

$$S_k = \left|\frac{s}{(r-\lambda)^2}\right| \left[1 + 2\left|\frac{s}{r-\lambda}\right| + 3\left|\frac{s}{r-\lambda}\right|^2 + \cdots + (k-1)\left|\frac{s}{r-\lambda}\right|^{k-2}\right]$$

$$+\left|\frac{1}{r-\lambda}\right|\left|1-\frac{s^k}{(r-\lambda)^k}\right|+\left(\sum_{n=0}^{\infty}(n+k)\left|\frac{s}{r-\lambda}\right|^n\right)$$
$$=\left|\frac{s}{(r-\lambda)^2}\right|M_k+\left|\frac{1}{r-\lambda}\right|N_k,$$

Where

$$M_k=1+2\left|\frac{s}{r-\lambda}\right|+3\left|\frac{s}{r-\lambda}\right|^2+\cdots+(k-1)\left|\frac{s}{r-\lambda}\right|^{k-2}$$
$$M_k=\sum_{n=0}^{k-2}(n+k)\left|\frac{s}{r-\lambda}\right|^n$$
$$N_k=\left|1-\frac{s^k}{(r-\lambda)^k}\right|\left(\sum_{n=0}^{\infty}(n+k)\left|\frac{s}{r-\lambda}\right|^n\right)$$

Clearly

$$\sup_k\frac{1}{k}M_k<\infty$$

Further

$$\frac{1}{k}N_k=\left|1-\left(\frac{s}{r-\lambda}\right)^k\right|\left(\frac{1}{k}\sum_{n=0}^{\infty}n\left|\frac{s}{r-\lambda}\right|^n+\sum_{n=0}^{\infty}n\left|\frac{s}{r-\lambda}\right|^n\right)$$

Then $\sup_k\frac{1}{k}N_k<\infty$ since $\left|\frac{s}{r-\lambda}\right|<1$, and so

$$\sup_k\frac{1}{k}S_k=\sup_k\frac{1}{k}\sum_{n=0}^{\infty}n\left|\sum_{v=1}^{k}(c_{nv}-c_{n+1,v})\right|$$
$$=\left|\frac{s}{(r-\lambda)^2}\right|\sup_k\frac{1}{k}M_k+\left|\frac{1}{r-\lambda}\right|\sup_k\frac{1}{k}N_k<\infty$$

This implies that $\lambda\notin\delta(B(r,s),h).\, thus$

$$\delta(B(r,s),h)\subseteq\{\lambda\in\mathbb{C}:|\lambda-r|\le|s|\}$$

Conversel, let $\lambda\notin\delta(B(r,s),h)$ then $(B(r,s)-\lambda I)^{-1}\in B(h)$ Since $(B(r,s)-\lambda I)^{-1}$ Transform of the unit sequence $e_0=(1,0,0,\dots)$ is in h that is

$$(B(r,s)-\lambda I)^{-1}e_0=\left(\frac{1}{r-\lambda},\frac{s}{(r-\lambda)^2},\frac{s^2}{(r-\lambda)^3},\dots.\right)\in h$$

Then $\lambda\neq r$ and $|\,(s/(r-\lambda))\,|\le 1$. Therefore, $\lambda\notin\{\lambda\in\mathbb{C}:|\,\lambda-r\,|<|s|\}$. Thus

$$\{\lambda \in \mathbb{C}: |\lambda - r| < |s|\} \subseteq \sigma(B(r,s),h).$$

Since the spectrum of any bounded linear operator is compact, we have

$$\{\lambda \in \mathbb{C}: |\lambda - r| \leq |s|\} \subseteq \sigma(B(r,s),h).$$

This completes the proof. ■

Theorem 2.3. The operator $B(r,s)$ has no eigenvalues in h.

Proof. Suppose $B(r,s)x = \lambda x$ for $x = (x = (x_k)_{k=1}^{\infty} \in h$, $x \neq \theta$. Then by solving the system of equations

$$\begin{aligned} rx_1 &= \lambda x_1 \\ sx_1 + rx_2 &= \lambda x_2 \\ sx_2 + rx_3 &= \lambda x_3 \\ &\vdots \\ sx_k + rx_{k+1} &= \lambda x_{k+1} \\ &\vdots \end{aligned}$$

we obtain

$$(r-\lambda)x_1 = 0 \text{ and } sx_k + (r-\lambda)x_{k+1} = 0,\ k \geq 1.$$

If $\lambda = r$, we have $x = \theta$. Also, if $\lambda \neq r$, we would have $x = \theta$. So $\sigma_p(B(r,s),h) = \emptyset$. ■

Theorem 2.4. The point spectrum of the adjoint operator $B(r,s)^*$ on h^* is given by

$$\sigma_p(B(r,s)^*,h^*) = \{\lambda \in \mathbb{C}: |\lambda - r| \leq |s|\}.$$

Proof. Suppose that $B(r,s)^* f = \lambda f$ for $f = (f_1, f_2, f_3, \dots) \neq \theta$ in $h^* \cong \sigma_\infty$. Then by solving the system of equations

$$\begin{aligned} (r-\lambda)f_1 &= -sf_2 \\ (r-\lambda)f_2 &= -sf_3 \\ &\vdots \\ (r-\lambda)f_k &= -sf_{k+1} \end{aligned}$$

we obtain

$$f_{k+1} = \left(\frac{\lambda-r}{s}\right)^k f_1,\ k = 1,2,\cdots.$$

Therefore, we must take $f_1 \neq 0$ since otherwise we would have $f = \theta$. Also, it is clear that $r \in \sigma_p(B(r,s)^*, h^*)$. On the other hand, if $\lambda \neq r$, then $f_k \neq 0$ for all $k \in \mathbb{N}$ and

$$\sup_n \frac{1}{n+1}\left|\sum_{k=1}^{n} f_k\right| = \sup_n \frac{1}{n+1}\left|\sum_{k=1}^{n}\left(\frac{\lambda-r}{s}\right)^{k-1} f_1\right|$$

$$= |f_1| \sup_n \left(\frac{1}{n+1}\right)\left|\sum_{k=1}^{n}\left(\frac{\lambda-r}{s}\right)^{k-1}\right|$$

$$\leq |f_1| \sup_n \frac{1}{n+1}\left[1 + \left|\frac{\lambda-r}{s}\right| + \left|\frac{\lambda-r}{s}\right|^2 + \ldots + \left|\frac{\lambda-r}{s}\right|^{n-1}\right],$$

which is finite for all $\lambda \in \{\lambda \in \mathbb{C}: |\lambda - r| \leq |s|\}$. Thus

$$\{\lambda \in \mathbb{C}: |\lambda - r| \leq |s|\} \subseteq \sigma_p(B(r,s)^*, h^*).$$

The second inclusion follows immediately from the fact that

$$\sigma_p(B(r,s)^*, h^*) \subseteq \sigma(B(r,s)^*, h^*) = \sigma(B(r,s), h).$$

This completes the proof. ■

Lemma 2.1. [25, Page 59]. If T is a bounded linear operator on a normed space X into a normed space Y, then T has a dense range in Y if and only if $(T^*)^{-1}$ exists.

Lemma 2.2. [25, Page 60]. T has a bounded inverse if and only if T^* is onto.

Lemma 2.3. [11, pp. 20; 14, pp. 38]. If T is a linear operator on a complex normed space X into itself, then $III_1(T, X)$ is an open set.

Lemma 2.4. If T is a bounded linear operator on a Banach space X into itself, then

$$\sigma_r(B(r,s), h) = \sigma_p(B(r,s)^*, h^*) \backslash \sigma_p(B(r,s), h).$$

Proof. For $\lambda \in \sigma_p(T^*, X^*) \backslash \sigma_p(T, X)$, the operator $T - \lambda I$ is one to one and hence has an inverse. But $T^* - \lambda I$ is not one to one. Now, Lemma 2.1 yields the fact that the range of the operator $T - \lambda I$ is not dense in X. This implies that $\lambda \in \sigma_r(B(r,s), h)$. ■

Theorem 2.5. The residual spectrum of the operator $B(r,s)$ on h is given by

$$\sigma_r(B(r,s),h) = \{\lambda \in \mathbb{C}: |\lambda - r| \leq |s|\}.$$

Proof. From Lemma 2.4 and Theorems 2.3 and 2.4, we obtain the desired consequence. ■

Theorem 2.6. The continuous spectrum of the operator $B(r,s)$ on h is

$$\sigma_c(B(r,s),h) = \emptyset.$$

Proof. Since $\sigma(B(r,s),h)$ is the union of the disjoint sets $\sigma_p(B(r,s),h)$, $\sigma_r(B(r,s),h)$and $\sigma_c(B(r,s),h)$, then Theorems 2.2, 2.3 and 2.5 imply $\sigma_c(B(r,s),h) = \emptyset$. ■

Next, we investigate the fine structure of the spectrum of the operator $B(r,s)$ with respect to the other classification schemes.

Indeed, for the operator $B(r,s)$ on the Hahn space h, we have

$$I_3(B(r,s),h) = II_3(B(r,s),h) = III_3(B(r,s),h) = \emptyset,$$

since

$$\sigma_p(B(r,s),h).$$

Also

$$II_2(B(r,s),h) = \emptyset,$$

since

$$\sigma_c(B(r,s),h) = \emptyset.$$

Moreover

$$I_2(B(r,s),h) = \emptyset,$$

by the closed graph theorem. Also

$$III_1(B(r,s),h) \cup III_2(B(r,s),h) = \sigma_r(B(r,s),h) = \{\lambda \in \mathbb{C}: |\lambda - r| \leq |s|\}. \qquad \textbf{2.2}$$

Next, we completely determine the parts $III_1(B(r,s),h)$ and $III_2(B(r,s),h)$. This gives a finer subdivision of the spectrum.

Theorem 2.7. The following statements hold:
1. $III_1(B(r,s),h) = \{\lambda \in \mathbb{C}: |\lambda - r| < |s|\}$,
2. $III_2(B(r,s),h) = \{\lambda \in \mathbb{C}: |\lambda - r| = |s|\}$.

Proof. (i) *Let* $\lambda \in \{\lambda \in \mathbb{C}: |\lambda - r| < |s|\}$. Then $\lambda \in \sigma_p(B(r,s)^*, h^*)$ by Theorem 2.4, that is, $(B(r,s)^* - \lambda I)^{-1}$ does not exist. So $B(r,s)^* - \lambda I$ is injective, which implies by Lemma 2.1 that, $B(r,s) - \lambda I$ has not a dense range; $\overline{R(B(r,s)-\lambda I)} \neq h$. Also, $\lambda \notin \sigma_p(B(r,s),h)$ by Theorem 2.3. Hence $B(r,s)-\lambda I$ has an inverse. Next, we must prove that $(B(r,s)-\lambda I)^{-1}$ is bounded, it suffices to show that $B(r,s)^*-\lambda I$ is onto, and then we use Lemma 2.2. For this purpose, given $y=(y_k)_{k=0}^{\infty} \in h^* \cong \sigma_\infty$, we must find $x=(x_k)_{k=0}^{\infty} \in \sigma_\infty$ such that $(B(r,s)^*-\lambda I)x=y$. Direct calculations show that

$$(r-\lambda)x_k + s x_{k+1} = y_k, \text{ for all } k \in \mathbb{N}.$$

Then we have

$$x_1 = \frac{1}{s}y0 + \frac{(\lambda-r)}{s}x_0$$

$$x_2 = \frac{1}{s}y1 + \frac{(\lambda-r)}{s^2}y0 + \frac{(\lambda-r)^2}{s^2}x0$$

$$x_3 = \frac{1}{s}y2 + \frac{(\lambda-r)}{s^2}y1 + \frac{(\lambda-r)^2}{s^2}y0 + \frac{(\lambda-r)^3}{s^3}x0$$

$$\vdots$$

$$x_k = \frac{1}{s}y0 + \frac{(\lambda-r)}{s^2}yk-2 + \frac{(\lambda-r)^{k-3}}{s^{k-2}}y2 + \frac{(\lambda-r)^{k-2}}{s^{k-1}}y1 + \frac{(\lambda-r)^{k-1}}{s^k}y0 + \frac{(\lambda-r)^k}{s^k}x0$$

So. We obtain

$$\sum_{k=0}^{n} xk = \frac{1}{s}y0\left(\frac{\lambda-r}{s}\right)^{n-1} + \frac{1}{s}(y0+y1)\left(\frac{\lambda-r}{s}\right)^{n-2} + \frac{1}{s}(y0+y1+y2)\left(\frac{\lambda-r}{s}\right)^{n-3} + \cdots + \frac{1}{s}(y0+y1+\cdots+y_{n-2})\left(\frac{\lambda-r}{s}\right) + \frac{1}{s}(y0+y1+\cdots+y_{n-1}) + \left|1+\frac{\lambda-r}{s}+\frac{(\lambda-r)^2}{s^2}+\cdots+\frac{(\lambda-r)^n}{s^n}\right|x0$$

Then

$$\left|\sum_{k=0}^{n} xk\right| \leq \frac{1}{|s|}|y0| + \left|\frac{\lambda - r}{s}\right|^{n-1} + \frac{1}{|s|}|y0 + y1|\left|\frac{\lambda - r}{s}\right|^{n-2}$$
$$+ \frac{1}{|s|}|y0 + y1 + y2|\left|\frac{\lambda - r}{s}\right|^{n-3} + \cdots$$
$$+ \frac{1}{|s|}|y0 + y1 + \cdots + yn - 2|\left|\frac{\lambda - r}{s}\right|$$
$$+ \frac{1}{|s|}|y0 + y1 + \cdots + yn - 1|$$
$$+ \left[1 + \left|\frac{\lambda - r}{s}\right| + \left|\frac{\lambda - r}{s}\right|^2 + \cdots + \left|\frac{\lambda - r}{s}\right|^n\right]|x0|$$

Since $\sum_{k=0}^{\infty}\left|\frac{\lambda-r}{s}\right|^k$ and $\sup_n \frac{1}{n+1}\sum_{k=0}^{n}\left|\frac{\lambda-r}{s}\right|^k$ are finite for all $\lambda \in \{\lambda \in \mathbb{C}: |\lambda - r| < |s|\}$, then $\sup_n \frac{1}{n+1}|\sum_{k=0}^{n} x_k| < \infty$. That is $x = (x_k)_{k=0}^{\infty} \in \sigma_\infty$. Therefore $B(r,s)^* - \lambda I$ is onto, and so, we conclude that $\{\lambda \in \mathbb{C}: |\lambda - r| < |s|\} \subseteq III_1(B(r,s),h)$. Further, by Lemma 2.3, we have

$III_1(B(r,s),h) \subseteq int(\{\lambda \in \mathbb{C}: |\lambda - r| \leq |s|\}) = \{\lambda \in \mathbb{C}: |\lambda - r| < |s|\}$.

(2) Follows immediately from Eq. (2.2).
This completes the proof. ■

The relations given in Eq. (1.2) and Proposition 1.1(e) imply the next theorem.

Theorem 2.8. The following statements hold:
(i) $\sigma_{ap}(B(r,s),h) = \{\lambda \in \mathbb{C}: |\lambda - r| = |s|\}$,
(ii) $\sigma_{co}(B(r,s),h) = \{\lambda \in \mathbb{C}: |\lambda - r| \leq |s|\}$,
(iii) $\sigma_\delta(B(r,s),h) = \{\lambda \in \mathbb{C}: |\lambda - r| \leq |s|\}$.

Now, we review the results concerning the spectra of the difference operator Δ on the Hahn sequence space [27], which are so related to our problem.

We will show by introducing an example that some statements of the following theorem given in [27] are incorrect.

Theorem 2.9. [27]. The following statements hold:

(i) $\sigma(\Delta, h) = \{\lambda \in \mathbb{C}: |\lambda - 1| \leq 1\}$,

(ii) $\sigma_p(\Delta, h) = \emptyset$,

(iii) $\sigma_p(\Delta^*, h^*) = \{\lambda \in \mathbb{C}: |\lambda - 1| < 1\}$,

(iv) $\sigma_r(\Delta, h) = \{\lambda \in \mathbb{C}: |\lambda - 1| < 1\}$,

(v) $\sigma_c(\Delta, h) = \{\lambda \in \mathbb{C}: |\lambda - 1| = 1\}$,

(vi) $\sigma_{ap}(\Delta, h) = \{\lambda \in \mathbb{C}: |\lambda - 1| \leq 1\}$,

(vii) $\sigma_{co}(\Delta, h) = \{\lambda \in \mathbb{C}: |\lambda - 1| < 1\}$,

(viii) $\sigma_\delta(\Delta, h) = \{\lambda \in \mathbb{C}: |\lambda - 1| \leq 1\}$.

Firstly, we prove that $\lambda = 2 \in \sigma_p(\Delta^*, h^*)$. Indeed, for $\theta \neq x = (x_k)_{k=0}^{\infty} \in \sigma_\infty$ with $\Delta^* x = 2x$, we have

$$\begin{aligned} x_0 - x_1 &= 2x_0 \\ x_1 - x_2 &= 2x_1 \\ &\vdots \end{aligned}$$

Therefore $x_n = (-1)^n x_0$. If $x_0 \neq 0$, so $x \neq \theta$ and

$$\sup_n (1/(n+1)) \left|\sum_{k=0}^{n} (-1)^k x_0\right| = |x_0| \sup_n (1/(n+1)) \left|\sum_{k=0}^{n} (-1)^k\right| < \infty.$$

Then $x \in \sigma_\infty$.

This proves that the statement (iii), and consequently the statements given by (iv),(v),(vi) and (vii) in this theorem are incorrect. Applying our results in Theorems 2.4, 2.5, 2.6 and 2.8, we obtain

$$\begin{aligned} \sigma_p(\Delta^*, h^*) &= \{\lambda \in C: |\lambda - 1| \leq 1\}, \\ \sigma_r(\Delta, h) &= \{\lambda \in \mathbb{C}: |\lambda - 1| \leq 1\}, \\ \sigma_c(\Delta, h) &= \emptyset, \\ \sigma_{ap}(\Delta, h) &= \{\lambda \in \mathbb{C}: |\lambda - 1| = 1\}, \\ \sigma_{co}(\Delta, h) &= \{\lambda \in \mathbb{C}: |\lambda - 1| \leq 1\}. \end{aligned}$$

This completes the proof. ■

Conclusion

Refer to the literature, El-Shabrawy and Abu-Janah [9] gave results regarding the fine spectrum in general to study the problem on the sequence spaces bv_0 and h without any detailed proofs about the spectra of the operator Δ_{ab} over h. In [9], They determined the spectra of the operator $B(r,s)$ on bv_0 without any details about $B(r,s)$ on h. Our results are more general than the corresponding results in the existing literature [4,7,8,10] and there are improvements in some proofs of the results.

References

[1] A. M. Akhmedov, F. Başar, The fine spectra of the difference operator Δ over the sequence space $\ell_p, (1 \leq p < \infty)$, Demonstratio Math. 39 (3) (2006), 585-595.

[2] A. M. Akhmedov, F. Başar, The fine spectra of the difference operator Δ over the sequence space bv_p, $(1 \leq p < \infty)$, Acta Math. Sin. (Engl. Ser.) 23 (10) (2007), 1757-1768.

[3] B. Altay, F. Başar, On the fine spectrum of the difference operator Δ on c_0 and c, Inform. Sci. 168 (2004), 217-224.

[4] B. Altay, F. Başar, On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces c_0 and c, Int. J. Math. Sci. 18 (2005), 3005-3013.

[5] J. Appell, E. De Pascale, A. Vignoli, Nonlinear Spectral Theory; de Gruyter Series in Nonlinear Analysis and Applications 10, Walter de Gruyter, Berlin, 2004.

[6] F. Başar, N. Durna, M. Yildirim, Subdivisions of the spectra for the generalized difference operator over certain sequence spaces, Thai J. Math. 9 (2) (2011), 279-289.

[7] H. Bilgiç, H. Furkan, On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces ℓ_p and bv_p, $(1 < p < \infty)$, Nonlinear Anal. 68 (2008), 499-506.

[8] A. J. Dutta, B. C. Tripathy, Fine spectrum of the generalized difference operator $B(r,s)$ over the class of convergent series, International Journal of Analysis, 2013, Art. ID 630436, 4 pp.

[9] S. R. El-Shabrawy, S. H. Abu-Janah, Spectra of the generalized difference operator on the sequence spaces bv_0 and h, Linear and Multilinear Algebra, dol:10.1080/03081087.1369492, accepted manuscript, 2017, 1-18.

[10] H. Furkan, H. Bilgiç, K. Kayaduman, On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces ℓ_1 and bv, Hokkaido Math. J. 35 (2006), 893-904.

[11] H. A. Gindler, A. E. Taylor, The minimum modulus of a linear operator and its use in spectral theory, Studia Math. 22 (1962), 15-41.

[12] G. Goes, S. Goes, Sequences of bounded variation and sequences of Fourier coefficients I, Math. Z. 118 (1970), 93-102.
[13] H. Hahn, Ūber Folgen linear operationen, Monatsh. Math. 32 (1922), 3-88.
[14] C. J. A. Halberg, Jr., A. Samuelsson, On the fine structure of spectra, Math. Scand. 29 (1971), 37-49.
[15] K. Kayaduman, H. Furkan, The fine spectra of the difference operator Δ over the sequence spaces ℓ_1 and bv, Int. Math. Forum 1 (24), (2006), 1153-1160.
[16] M. Kirişci, A survey on the Hahn sequence spaces, Gen. Math. Notes 19 (2) (2013), 37-58.
[17] M. Kirişci, The Hahn sequence spaces defined by Cesàro mean, Abstr. Appl. Anal., 2013, Art. ID 817659, 6 pp.
[18] E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley & Sons Inc., New York, Chichester, Brisbane, Toronto, 1978.
[19] I. J. Maddox, Elements of Functional Analysis, Cambridge Univ. Press, London, 1970.
[20] W. C. Rao, The Hahn sequence spaces I, Bull. Calcutta Math. Soc. 82 (1990), 72-78.
[21] W. C. Rao, T. G. Srinivasalu, The Hahn sequence spaces II, Y.Y.U. J. Fac. Edu. 2 (1996), 43-45.
[22] W. C. Rao, N. Subramanian, The Hahn sequence spaces III, Bull. Malaysian Math. Sci. Soc. 25 (2002), 163-171.
[23] M. H. Stone, Linear Transformations in Hilbert Space and their Applications to Analysis, American Mathematical Society, New York, 1932.
[24] A. E. Taylor, Introduction to Functional Analysis, John Wiley & Sons, Inc., New York, 1958.
[25] A. E. Taylor, C. J. A. Halberg, Jr., General theorems about a bounded linear operator and its conjugate, J. Reine Angew. Math. 198 (1957), 93-111.
[26] A. Wilansky, Summability Through Functional Analysis, North-Holland Mathematics Studies, vol. 85, Amsterdam, New York, Oxford, 1984.
[27] M. Yeşilkayagil, M. Kirişci, On the spectrum of the forward difference operator on the Hahn space, Gen. Math. Notes 33 (2) (2016), 1-16.

مجـلة الـتربـوي
Journal of Educational
ISSN: 2011- 421X
Arcif Q3

معامل التأثير العربي 1.5
العدد 20

الفهـــــــــرس