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بحوث العدد

- Solution of some problems of linear plane elasticity in doubly-connected regions by the method of boundary integrals
- Common Fixed-Point Theorems for Occasionally Weakly Compatible Mappings in Fuzzy 2-Metric Space
- THE STARLIKENESS AND CONVEXITY OF P-VALENT FUNCTIONS INVOLVING CERTAIN FRACTIONAL DERIVATIVE OPERATOR
- Utilizing Project-Based Approach in Teaching English through Information Technology and Network Support
- An Acoustic Study of Voice Onset Time in Libyan Arabic

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Abstract.
The object of the present paper is to derive some inequalities for the starlikeness and convexity of analytic and p-valent functions in the open unit disk involving certain fractional derivative operator. Some interesting consequences of the main result are also mentioned.
Key words and phrases: p-valent function, starlike function, convex function, fractional derivative operator, Jack's Lemma.
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1. Introduction and Preliminaries

Let $A(p)$ denote the class of functions defined by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}\, z^{p+n} \quad , \qquad a_{p+n} \in R;\ p \in N \tag{1.1}$$

which are analytic and p-valent in the open unit disk $\mathcal{U} = \{z: |z| < 1\}$, and set $A(1) \equiv A$. A function $f(z) \in A(p)$ is called p-valent starlike of order α if $f(z)$ satisfies the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{1.2}$$

for $0 \le \alpha < p,\ p \in N$ and $z \in \mathcal{U}$. We denote by $S^*(p,\alpha)$ the class of all p-valent starlike functions of order α. Also a function $f(z) \in A(p)$ is called p-valent convex of order α if $f(z)$ satisfies the condition

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \tag{1.3}$$

for $0 \le \alpha < p,\ p \in N$ and $z \in \mathcal{U}$. We denote by $K(p,\alpha)$ the class of all p-valent convex functions of order α. We note that

$$f(z) \in K(p,\alpha) \Leftrightarrow \frac{zf'(z)}{p} \in S^*(p,\alpha) \tag{1.4}$$

for $0 \le \alpha < p$.

The class $S^*(p,\alpha)$ was introduced by Patil and Thakare [10], and the class $K(p,\alpha)$ was introduced by Owa [9].

Let ${}_2F_1(a,b;c;z)$ be the Gauss hypergeometric function defined for $z \in \mathcal{U}$ by, (see Srivastava and Karlsson [12])

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n\, z^n}{(c)_n\, n!} \tag{1.5}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & , \quad n=0 \\ \lambda(\lambda+1)(\lambda+2)\ldots.(\lambda+n-1) & , \quad n \in N \end{cases} \tag{1.6}$$

for $\lambda \neq 0, -1, -2, \ldots$

We recall the following definitions of fractional derivative operators which were used by Owa [8], (see also [11]) as follows:

Definition 1.1. The fractional derivative operator of order λ is defined,

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi \tag{1.7}$$

where $0 \leq \lambda < 1$, $f(z)$ is analytic function in a simply- connected region of the z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring log $(z-\xi)$ to be real when $z-\xi > 0$.

Definition 1.2. Let $0 \leq \lambda < 1$, and $\mu, \eta \in R$. Then, in terms of the familiar Gauss's hypergeometric function ${}_2F_1$, the generalized fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}$ is

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{d}{dz}\left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)} \int_0^z (z-\xi)^{-\lambda} f(\xi)\, {}_2F_1\left(\mu-\lambda, 1-\eta; 1-\lambda; 1-\frac{\xi}{z}\right) d\xi\right) \tag{1.8}$$

where $f(z)$ is analytic function in a simply- connected region of the z-plane containing the origin with the order $f(z) = O(|z|^{\varepsilon})$, $z \to 0$, where $\varepsilon > \max\{0, \mu-\eta\} - 1$, and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 1.3. Under the hypotheses of Definition 1.2, the fractional derivative operator $J_{0,z}^{\lambda+m,\mu+m,\eta+m} f(z)$ of a function $f(z)$ is defined by

$$J_{0,z}^{\lambda+m,\mu+m,\eta+m} f(z) = \frac{d^m}{dz^m} J_{0,z}^{\lambda,\mu,\eta} f(z) \tag{1.9}$$

Notice that

$$J_{0,z}^{\lambda,\lambda,\eta} f(z) = D_z^{\lambda} f(z), \qquad 0 \leq \lambda < 1 \tag{1.10}$$

With the aid of the above definitions, we define a modification of the fractional derivative
operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$ by

$$M_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\eta)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)} z^{\mu} J_{0,z}^{\lambda,\mu,\eta} f(z) \tag{1.11}$$

for $f(z) \in A(p)$ and $\lambda \geq 0$; $\mu < p+1$; $\eta > \max(\lambda, \mu) - p - 1; p \in N$. Then it is observed that $M_{0,z}^{\lambda,\mu,\eta} f(z)$ maps $A(p)$ onto itself as follows:

$$M_{0,z}^{\lambda,\mu,\eta} f(z) = z^p + \sum_{n=1}^{\infty} \delta_n(\lambda,\mu,\eta,p)\, a_{p+n} z^{p+n} \tag{1.12}$$

where

$$\delta_n(\lambda,\mu,\eta,p) = \frac{(p+1)_n (p+1-\mu+\eta)_n}{(p+1-\mu)_n (p+1-\lambda+\eta)_n} \tag{1.13}$$

It is easily verified from (1.12) that

$$z\left(M_{0,z}^{\lambda,\mu,\eta} f(z)\right)' = (p-\mu)\, M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z) + \mu\, M_{0,z}^{\lambda,\mu,\eta} f(z) \quad (1.14)$$

This identity plays a critical role in obtaining information about functions defined by use of the fractional derivative operator. Our results in this paper will rely heavily on the identity.

Notice that

$$M_{0,z}^{0,0,\eta} f(z) = f(z),$$

and

$$M_{0,z}^{1,1,\eta} f(z) = \frac{zf'(z)}{p}$$

Making use of the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta} f(z)$, we now introduce an interesting generalization of the class $S^*(p,\alpha)$ of functions in $A(p)$ which satisfy the inequality (1.2).

Definition 1.4. A function $f(z) \in A(p)$ is said to be in the subclass $S_{\lambda,\mu,\eta}(p,\alpha)$ if it satisfies the inequality

$$\operatorname{Re}\left\{(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right\} > \alpha \ , \qquad z \in \mathcal{U} \tag{1.15}$$

$$(0 \le \alpha < p \ ; \ \lambda \ge 0, \mu < p+1 \ ; \ \eta > \max(\lambda,\mu) - p - 1 \ ; \ p \in N)$$

Also, a function $f(z) \in A(p)$ is said to be in the subclass $K_{\lambda,\mu,\eta}(p,\alpha)$ iff

$$\frac{zf'(z)}{p} \in S_{\lambda,\mu,\eta}(p,\alpha)$$

Observe that, $S_{0,0,\eta}(p,\alpha) = S^*(p,\alpha)$ and $K_{0,0,\eta}(p,\alpha) = K(p,\alpha)$.

There are many papers in which various sufficient conditions for multivalent starlikeness have been obtained [1, 2, 3, 4, 6, 7]. In this paper we derive new sufficient conditions for the operator $M_{0,z}^{\lambda,\mu,\eta}f(z)$ to be p-valently starlike and p-valently convex in $\mathcal{U}$. For their proofs we used Jack's Lemma (Lemma 1.5) below. Some interesting corollaries are also deduced from our main results.

To establish our results, we shall need the following lemma.

Lemma 1.5. [5] Let $w(z)$ be non-constant and analytic function in $\mathcal{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r, (0 < r < 1)$ at the point z_0, then $z_0 w'(z_0) = c w(z_0)$, where $c \ge 1$.

2. The main results

By using Lemma 1.5, we now prove the following result.

Theorem 2.1. Let $z \in \mathcal{U}$; $0 \le \alpha < p$; $\lambda \ge 0$; $\mu < p+1$; $\eta > \max(\lambda,\mu) - p - 1$ and $f(z) \in A(p)$ and if $M_{0,z}^{\lambda,\mu,\eta}f(z)$ satisfies anyone of the following inequalities:

$$\left|\frac{1+(p-\mu-1)\dfrac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}-(p-\mu)}{(p-\mu)\dfrac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}-(p-\mu)}-1\right| < \frac{1}{2p-\mu-\alpha} \tag{2.1}$$

$$\left|1+(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}-(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right| < \frac{p-\alpha}{2p-\mu-\alpha} \tag{2.2}$$

$$\left|\frac{M_{0,z}^{\lambda,\mu,\eta}f(z)}{(p-\mu)\,M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}\left(1+(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}\right)-1\right| < \frac{p-\alpha}{(2p-\mu-\alpha)^2} \qquad (2.3)$$

$$\left|(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\left(1+(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)} - (p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right)\right| < (p-\alpha) \qquad (2.4)$$

$$\mathrm{Re}\left\{(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\left(\frac{1+(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}-(p-\mu)}{(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}-(p-\mu)} - 1\right)\right\} < 1 \qquad (2.5)$$

Then $M_{0,z}^{\lambda,\mu,\eta}f(z) \in S_{\lambda,\mu,\eta}(p,\alpha)$.

Proof. Let $f(z) \in A(p)$. Since

$$(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)} = (p-\mu)+d_1z+d_2z^2+\cdots, \quad z\in\mathcal{U}$$

Define the function $w(z)$ by

$$(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)} = (p-\mu)+(p-\alpha)w(z), \quad z\in\mathcal{U};\ 0\le\alpha < p \qquad (2.6)$$

It is clear that $w(z)$ is analytic in $\mathcal{U}$ with $w(0)=0$. Also, we can find from (2.6) that

$$\frac{z\left(M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)\right)'}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)} - \frac{z\left(M_{0,z}^{\lambda,\mu,\eta}f(z)\right)'}{M_{0,z}^{\lambda,\mu,\eta}f(z)} = \frac{(p-\alpha)zw'(z)}{(p-\mu)+(p-\alpha)w(z)} \qquad (2.7)$$

By using (1.14) to (2.7), we have

$$(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}-(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}+1 =\frac{(p-\alpha)zw'(z)}{(p-\mu)+(p-\alpha)w(z)} \quad (2.8)$$

It follows from (2.6) that

$$1+(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}-(p-\mu)= (p-\alpha)w(z)\left(1+\frac{zw'^{(z)}}{w(z)}\frac{1}{(p-\mu)+(p-\alpha)w(z)}\right) \quad (2.9)$$

Hence,

$$F_1(z)=\frac{1+(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}-(p-\mu)}{(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}-(p-\mu)}-1$$

$$=\frac{zw'(z)}{w(z)}\frac{1}{(p-\mu)+(p-\alpha)w(z)} \quad (2.10)$$

$$F_2(z)=1+(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}-(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}$$

$$=\frac{(p-\alpha)zw'(z)}{(p-\mu)+(p-\alpha)w(z)} \quad (2.11)$$

$$F_3(z)=\frac{M_{0,z}^{\lambda,\mu,\eta}f(z)}{(p-\mu)\,M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}\left(1+(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}\right)-1$$

$$=\frac{(p-\alpha)zw'(z)}{[(p-\mu)+(p-\alpha)w(z)]^2} \quad (2.12)$$

$$F_4(z)=(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\left(1+(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)} -(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\right)$$

$$=(p-\alpha)zw'(z) \quad (2.13)$$

$$F_5(z) = (p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}\left(\frac{1+(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z)}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}-(p-\mu)}{(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z)}{M_{0,z}^{\lambda,\mu,\eta}f(z)}-(p-\mu)}-1\right)$$

$$=\frac{zw'(z)}{w(z)} \tag{2.14}$$

If there exist a point $z_0 \in \mathcal{U}$ such that

$$\max_{|z|\le|z_0|}|w(z)| = |w(z_0)| = 1$$

Then by Lemma 1.5, we have

$$z_0w'(z_0) = cw(z_0), \quad c \ge 1$$

Therefore, the equations (2.10)-(2.14) yield,

$$|F_1(z_0)| = \left|\frac{z_0w'(z_0)}{w(z_0)}\frac{1}{(p-\mu)+(p-\alpha)w(z_0)}\right| = \frac{c|w(z_0)|}{|(p-\mu)+(p-\alpha)w(z_0)|}$$

$$\ge \frac{1}{2p-\mu-\alpha} \tag{2.15}$$

$$|F_2(z_0)| = \left|\frac{(p-\alpha)z_0w'(z_0)}{(p-\mu)+(p-\alpha)w(z_0)}\right| = \frac{c(p-\alpha)|w(z_0)|}{|(p-\mu)+(p-\alpha)w(z_0)|}$$

$$\ge \frac{(p-\alpha)}{2p-\mu-\alpha} \tag{2.16}$$

$$|F_3(z_0)| = \left|\frac{(p-\alpha)z_0w'(z_0)}{[(p-\mu)+(p-\alpha)w(z_0)]^2}\right| = \frac{c(p-\alpha)|w(z_0)|}{|(p-\mu)+(p-\alpha)w(z_0)|^2}$$

$$\ge \frac{(p-\alpha)}{(2p-\mu-\alpha)^2} \tag{2.17}$$

$$|F_4(z_0)| = |(p-\alpha)z_0w'(z_0)| = c(p-\alpha)|w(z_0)|$$

$$\ge p-\alpha \tag{2.18}$$

$$\operatorname{Re}\{F_5(z_0)\} = \operatorname{Re}\left\{\frac{z_0 w'(z_0)}{w(z_0)}\right\} = c \geq 1 \qquad (2.19)$$

which contradict our assumptions (2.1)-(2.5), respectively. Therefore, $|w(z)| < 1$ holds true for all $z \in \mathcal{U}$. From (2.6), we have

$$\left|(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)} - (p-\mu)\right| = (p-\alpha)|w(z)| < p-\alpha \qquad (2.20)$$

which implies that

$$\operatorname{Re}\left\{(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z)}{M_{0,z}^{\lambda,\mu,\eta} f(z)}\right\} > \alpha$$

and hence $M_{0,z}^{\lambda,\mu,\eta} f(z) \in S_{\lambda,\mu,\eta}(p,\alpha)$.

If we put $\frac{zf'(z)}{p}$ instead of $f(z)$ in Theorem 2.1, we then obtain the following theorem.

Theorem 2.2. Let $z \in \mathcal{U}$; $0 \leq \alpha < p$; $\lambda \geq 0$; $\mu < p+1$; $\eta > \max(\lambda,\mu) - p - 1$ and $f(z) \in A(p)$ and if $M_{0,z}^{\lambda,\mu,\eta}\left\{\frac{zf'(z)}{p}\right\}$ satisfies anyone of the following inequalities:

$$\left|\frac{1 + (p-\mu-1)\dfrac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}\left\{\frac{zf'(z)}{p}\right\}}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}\left\{\frac{zf'(z)}{p}\right\}} - (p-\mu)}{(p-\mu)\dfrac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}\left\{\frac{zf'(z)}{p}\right\}}{M_{0,z}^{\lambda,\mu,\eta}\left\{\frac{zf'(z)}{p}\right\}} - (p-\mu)} - 1\right| < \frac{1}{2p-\mu-\alpha} \qquad (2.21)$$

$$\left|1 + (p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}\left\{\frac{zf'(z)}{p}\right\}}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}\left\{\frac{zf'(z)}{p}\right\}} - (p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}\left\{\frac{zf'(z)}{p}\right\}}{M_{0,z}^{\lambda,\mu,\eta}\left\{\frac{zf'(z)}{p}\right\}}\right| < \frac{p-\alpha}{2p-\mu-\alpha} \qquad (2.22)$$

$$\left|\frac{M_{0,z}^{\lambda,\mu,\eta}\left\{\frac{zf'(z)}{p}\right\}}{(p-\mu)\,M_{0,z}^{\lambda+1,\mu+1,\eta+1}\left\{\frac{zf'(z)}{p}\right\}}\left(1 + (p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}\left\{\frac{zf'(z)}{p}\right\}}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}\left\{\frac{zf'(z)}{p}\right\}}\right) - 1\right| < \frac{p-\alpha}{(2p-\mu-\alpha)^2} \qquad (2.23)$$

$$\left|(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}\left\{\frac{zf'(z)}{p}\right\}}{M_{0,z}^{\lambda,\mu,\eta}\left\{\frac{zf'(z)}{p}\right\}}\left(1+(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}\left\{\frac{zf'(z)}{p}\right\}}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}\left\{\frac{zf'(z)}{p}\right\}} - (p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}\left\{\frac{zf'(z)}{p}\right\}}{M_{0,z}^{\lambda,\mu,\eta}\left\{\frac{zf'(z)}{p}\right\}}\right)\right| < (p-\alpha) \quad (2.24)$$

$$\operatorname{Re}\left\{(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}\left\{\frac{zf'(z)}{p}\right\}}{M_{0,z}^{\lambda,\mu,\eta}\left\{\frac{zf'(z)}{p}\right\}}\left(\frac{1+(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}\left\{\frac{zf'(z)}{p}\right\}}{M_{0,z}^{\lambda+1,\mu+1,\eta+1}\left\{\frac{zf'(z)}{p}\right\}}-(p-\mu)}{(p-\mu)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}\left\{\frac{zf'(z)}{p}\right\}}{M_{0,z}^{\lambda,\mu,\eta}\left\{\frac{zf'(z)}{p}\right\}}-(p-\mu)} - 1\right)\right\} < 1 \quad (2.25)$$

Then $M_{0,z}^{\lambda,\mu,\eta}f(z) \in K_{\lambda,\mu,\eta}(p,\alpha)$.

By setting $\lambda=\mu=0$ in Theorem 2.1, we obtain the following result.

Corollary 2.3. Let $f(z) \in A(p)$; $z \in \mathcal{U}$; $0 \le \alpha < p$. If $f(z)$ satisfies anyone of the following inequalities:

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}-p}{\frac{zf'(z)}{f(z)}-p}-1\right| < \frac{1}{2p-\alpha} \quad (2.26)$$

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right|<\frac{p-\alpha}{2p-\alpha} \tag{2.27}$$

$$\left|\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right|<\frac{p-\alpha}{(2p-\alpha)^2} \tag{2.28}$$

$$\left|\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right|<p-\alpha \tag{2.29}$$

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left(\frac{1+\frac{zf''(z)}{f'(z)}-p}{\frac{zf'(z)}{f(z)}-p}-1\right)\right\}<1 \tag{2.30}$$

Then $f(z)\in S^*(p,\alpha)$.

If we put $p=1$ Corollary 2.3, we obtain the following result.

Corollary 2.4. Let $f(z)\in A$; $z\in\mathcal{U}$; $0\le\alpha<1$. If $f(z)$ satisfies anyone of the following inequalities:

$$\left|\frac{\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}-1}-1\right|<\frac{1}{2-\alpha} \tag{2.31}$$

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right|<\frac{1-\alpha}{2-\alpha} \tag{2.32}$$

$$\left|\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right|<\frac{1-\alpha}{(2-\alpha)^2} \tag{2.33}$$

$$\left|\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right|<1-\alpha \tag{2.34}$$

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left(\frac{\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}-1}-1\right)\right\}<1 \tag{2.35}$$

Then $f(z)\in S^*(\alpha)$.

By setting $\lambda=\mu=0$ in Theorem 2.2, we obtain the following result.

Corollary 2.5. Let $f(z)\in A(p)$; $z\in\mathcal{U}$; $0\le\alpha<p$. If $f(z)$ satisfies anyone of the following inequalities:

$$\left|\frac{1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}-p}{1+\frac{zf''(z)}{f'(z)}-p}-1\right|<\frac{1}{2p-\alpha} \tag{2.36}$$

$$\left|\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}-\frac{zf''(z)}{f'(z)}\right|<\frac{p-\alpha}{2p-\alpha} \tag{2.37}$$

$$\left|\frac{f'(z)}{f'(z)+zf''(z)}\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right)-1\right|<\frac{p-\alpha}{(2p-\alpha)^2} \tag{2.38}$$

$$\left|z\left(1+\frac{zf''(z)}{f'(z)}\right)\left(\frac{2f''(z)+zf'''(z)}{f'(z)+zf''(z)}-\frac{f''(z)}{f'(z)}\right)\right| < p-\alpha \qquad (2.39)$$

$$\mathrm{Re}\left\{\left(1+\frac{zf''(z)}{f'(z)}\right)\left(\frac{1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}-p}{1+\frac{zf''(z)}{f'(z)}-p}-1\right)\right\} < 1 \qquad (2.40)$$

Then $f(z) \in K(p,\alpha)$.

If we put $p=1$ Corollary 2.5, we obtain the following result.

Corollary 2.6. Let $f(z) \in A$; $z \in \mathcal{U}$; $0 \leq \alpha < 1$. If $f(z)$ satisfies anyone of the following inequalities:

$$\left|\frac{f'(z)}{f''(z)}\left(\frac{2f''(z)+zf'''(z)}{f'(z)+zf''(z)}\right)-1\right| < \frac{1}{2-\alpha} \qquad (2.41)$$

$$\left|\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}-\frac{zf''(z)}{f'(z)}\right| < \frac{1-\alpha}{2-\alpha} \qquad (2.42)$$

$$\left|\frac{f'(z)}{f'(z)+zf''(z)}\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right)-1\right| < \frac{1-\alpha}{(2-\alpha)^2} \qquad (2.43)$$

$$\left|z\left(1+\frac{zf''(z)}{f'(z)}\right)\left(\frac{2f''(z)+zf'''(z)}{f'(z)+zf''(z)}-\frac{f''(z)}{f'(z)}\right)\right| < 1-\alpha \qquad (2.44)$$

$$\mathrm{Re}\left\{\left(1+\frac{zf''(z)}{f'(z)}\right)\left(\frac{f'(z)[2f''(z)+zf'''(z)]}{f''(z)[f'(z)+zf''(z)]}-1\right)\right\} < 1 \qquad (2.45)$$

Then $f(z) \in K(\alpha)$.

Remark: The corollaries 2.3 - 2.6 correspond to the known results given by Irmak and Raina [[4], corollaries 1 - 4].

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