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هيئة التحرير

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أعضاء هيئة التحرير

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استشارات فنية وتصميم الغلاف: أ. حسين ميلاد أبو شعالة

المجلة ترحب بما يرد عليها من أبحاث وعلى استعداد لنشرها بعد التحكيم .
المجلة تحترم كل الاحترام آراء المحكمين وتعمل بمقتضاها .
كافة الآراء والأفكار المنشورة تعبر عن آراء أصحابها ولا تتحمل المجلة تبعاتها .
يتحمل الباحث مسؤولية الأمانة العلمية وهو المسؤول عما ينشر له .
البحوث المقدمة للنشر لا ترد لأصحابها نشرت أو لم تنشر .
حقوق الطبع محفوظة للكلية .

بحوث العدد

الافتتاحية

إن الفرد الناجح في حياته، وكذلك المجتمعات والدول هم الذين يحددون أهدافهم، ويضعون في حسبانهم تحقيقها، والوصول إليها، فإذا حدد الفرد والمجتمع لنفسه هدفا فلن يضيع في متاهات الحياة، وسوف يصل إلى المطلوب، فتحديد الهدف أمر مهم ومقوم من مقومات النجاح، لذا على الآباء والمربين، أن يعلموا الأبناء- ومنذ السنوات الأولى في دراستهم- أن يحددوا لأنفسهم أهدافا ينبغي عليهم الاجتهاد من أجل الوصول إليها وتحقيقها، كما يجب أن يعلموهم معايير الأهداف حتى تتوافق مع رغباتهم وقدراتهم.

وعلى المجتمع كله والدول في عالمنا العربي أن يضعوا أهدافا واضحة المعالم للنهوض بالمجتمع يعرفها الصغير قبل الكبير، والجاهل قبل المتعلم، فيسعى الجميع وتتضافر الجهود من أجل تحقيقها وتنفيذها، لا أن تكون طوباوية لا يشعر بها الأفراد، ولا يحسون بقيمتها، فلا يسعون ولا يتعاونون لتحقيقها، بل نجدهم في بعض الأحيان يعملون عكسها لعدم درايتهم بها.

ونتيجة لعدم وجود الأهداف الواضحة المعالم في مجتمعاتنا أفرادا وجماعات لم يصل الفرد منا- عقليا وفكريا واجتماعيا واقتصاديا- إلى مستوى المسؤولية؛ ولم تصل مجتمعاتنا إلى أولى درجات الرقي، فالملاحظ على شبابنا الإهمال والتسيب واللامباة نحو نفسه ونحو مجتمعه، فيقبل بأدنى المراتب ولم يعد في أنظارهم إلا أمرين: المال وبأي وجه كان، والمنصب المرموق دون السعي إلى مؤهلاته، فضعفت لديهم العزيمة، وخارت القوى، ووقع الكثير في سفاسف الأمور.

وفي المقابل نجد أن شبابا كانت أهدافهم واضحة، ومقاصدهم معروفة ارتقوا بفضل ذلك إلى مقامات مرموقة، ووصلوا إلى ما يطمحون إليه، مع شيء التشجيع والمتابعة، فمن سار الطريق وصل.

هيئة التحرير

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Abstract:

The main object of this paper is to introduce and study the new subclasses $TS_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$ and $TR_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$ of analytic functions with negative coefficients defined by a linear operator . Coefficients inequalities for functions belonging to these subclasses are determined. In addition, radii of close-to-convexity, starlikeness and convexity, closure theorems are obtained.

1. Introduction

Let $A(n)$ denote the class of all analytic functions in the open unit disc

$\mathrm{U}=\{z\in \mathrm{C}:|z|<1\}$, of the form:

$$f(z)=z+\sum_{k=n+1}^{\infty}a_k z^k, (n=1,2,3,...). \quad (1)$$

Denote $T(n)$ the subclass of $A(n)$ consisting of functions of the form:

$$f(z)=z-\sum_{k=n+1}^{\infty}a_k z^k, (a_k\geq 0, n\in \mathrm{N}). \quad (2)$$

For functions $f \in A(n)$ given by (1) and $g \in A(n)$ given by

$$g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k,$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{k=n+1}^{\infty} a_k b_k z^k.$$

If f , g are analytic in U, we say that f is subordinate to g , denoted by $f \prec g$, if there exists a function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathrm{U})$, such that $f(z) = g(w(z))$, $(z \in \mathrm{U})$. It is known that $f(z) \prec g(z) \quad (z \in \mathrm{U}) \Rightarrow f(0) = g(0) \quad and \quad f(\mathrm{U}) \subset g(\mathrm{U})$.

Let the function $\varphi(a,c;z)$ be given by

$$\varphi(a,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1}, \quad (z \in \mathrm{U}, c \neq 0,-1,-2,-3,\ldots),$$

where $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial).

Corresponding to the function $\varphi(a,c;z)$, Carlson and Shaffer [14] introduced a linear operator $L(a,c)$ by

$$L(a,c)f(z) := \varphi(a,c;z) * f(z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} a_k z^{k+1}.$$

Note that:

$L(a,a)$ is the identity operator,and $L(a,c) = L(a,b)L(b,c) \quad (b,c \neq 0,-1,\ldots)$.

The author [1, 3] has recently introduced a new linear operator $D_l^{m,\lambda}(a,b)f(z)$ as the following:

Definition 1.1 *Let*

$$\phi_l^{m,\lambda}(a,b;z)=\sum_{k=0}^{\infty}\left(1+\frac{\lambda k}{1+l}\right)^m\frac{(a)_k}{(b)_k}z^{k+1},$$

where $(z\in U, b\neq 0,-1,-2,-3,...), \lambda\geq 0, m\in Z, l\geq 0,$ and $(x)_k$ is the Pochhammer symbol.

We defines a linear operator $D_l^{m,\lambda}(a,b):A\to A$ by the following Hadamard product:

$$D_l^{m,\lambda}(a,b)f(z):=\phi_l^{m,\lambda}(a,b;z)*f(z)=\sum_{k=0}^{\infty}\left(1+\frac{\lambda k}{1+l}\right)^m\frac{(a)_k}{(b)_k}a_k z^{k+1}. \quad (3)$$

Note that:

$$D_0^{0,\lambda}(a,b)f(z)=L(a,b)f(z),$$

$$(1+l)D_l^{m,\lambda}(a,b)f(z)=(1-\lambda+l)L(a,b)f(z)+\lambda z(L(a,b)f(z))'=D_\lambda(L(a,b)f(z)), \lambda\geq 0,$$

$D_l^{m,\lambda}(a,b)f(z)=D_\lambda(D_l^{m-1,\lambda}(a,b)f(z)),$ where $m\in Z.$

Special cases of this operator includes:

- $D_0^{m,0}(a,b)f(z)=D_l^{0,\lambda}(a,b)f(z)=L(a,b)f(z).$
- the Ruscheweyh derivative operator [10] in the cases: $D_0^{0,0}(\beta+1,1)f(z)=D^{\beta}f(z);\beta\geq -1.$
- the Salagean derivative operator [12]: $D_0^{m,1}(1,1)f(z).$
- the generalized Salagean derivative operator introduced by Al-Oboudi [11]: $D_0^{m,\lambda}(1,1)f(z).$
- the Catas drivative operator [17]: $D_l^{m,\lambda}(1,1)f(z),$ and finally.
- The fractional operator introduced by Owa and Srivastava [18]

$$D_0^{0,0}(2,2-\gamma)f(z)=\Omega^{\gamma}f(z)=\Gamma(2-\gamma)z^{\gamma}D_z^{\gamma}f(z);$$

$D_z^{\gamma}f(z)$ is the fractional derivative of f of order $\gamma;\gamma\neq 2,3,4,\cdots.$

Now, we introduce new subclasses of analytic functions involving our operator $D_l^{m,\lambda}(a,b)$.

Definition 1.2 *A function* $f\in T(n)$ *is said to be in the subclass* $TS_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$, *for* $(0<\beta\leq 1,\quad \gamma\in \mathbb{C}/\{0\})$ *if and only if:*

$$\left|\frac{1}{\gamma}\left(\frac{zv'}{v}-1\right)\right|<\beta,$$

where

$$\frac{zv'}{v}=\frac{z(D_l^{m,\lambda}(a,b))'+\delta z^2(D_l^{m,\lambda}(a,b)f(z))''}{(1-\delta)D_l^{m,\lambda}(a,b)f(z)+\delta z(D_l^{m,\lambda}(a,b)f(z))'}, \tag{5}$$

$(z\in \mathrm{U},\quad 0\leq\delta\leq 1).$

Definition 1.3 *A function* $f\in T(n)$ *is said to be in the subclass* $TR_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$ *if and only if*

$$\left|\frac{1}{\gamma}(v'-1)\right|<\beta,$$

$(z\in \mathrm{U},\quad 0\leq\delta\leq 1,\quad 0<\beta\leq 1,\quad \gamma\in \mathbb{C}/\{0\}).$

We note that there are some known subclasses of $TS_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$ and $TR_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$

Remark 1.1 *(1) If* $m=0,\quad and\quad a=c=0$ *then*

$$TS_{\beta,\gamma}^{0,\delta}(\lambda,l,0,0)=S_n(\beta,\gamma,\delta).$$

(2) If $m=0,\quad and\quad a=c=0$ then

$$TR_{\beta,\gamma}^{0,\delta}(\lambda,l,0,0)=R_n(\beta,\gamma,\delta).$$

The classes $S_n(\beta,\gamma,\delta)$ and $R_n(\beta,\gamma,\delta)$ were investigated in [5].

(3)If $m=0,\quad a=c=0$ and $\delta=0,\quad \beta=|b|,\quad \gamma=1$ then

$$TS_{|b|,1}^{0,0}(0,l,\lambda,c)=S_1^*(b),$$

where $(b\in \mathrm{C}/\{0\})$. The class $S_1^*(b)$ was studied in [6].

2. Coefficient bounds

In this section, we obtain necessary and sufficient conditions for a function to be in the subclasses $TS_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$ and $TR_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$ respectively.

Theorem 2.1 *Let the function f be defined by (3). Then f belongs to the subclass $TS_l^{m,\lambda}(a,b)$ if and only if*

$$\sum_{k=n+1}^{\infty}[1+\delta(k-1)](k+\beta|\gamma|-1)\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m\frac{(a)_{k-1}}{(c)_{k-1}}\right|a_k\le\beta|\gamma|,$$

$(z\in \mathrm{U},\quad 0\le\delta\le1,\quad 0<\beta\le1,\quad \gamma\in \mathrm{C}/\{0\}).$

Proof: Suppose $f\in TS_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$. By making use of (4) we easily obtain

$$Re\left\{\frac{zv'}{v}-1\right\}>-\beta|\gamma|\quad(z\in \mathrm{U}),$$

which, in view of (5), gives :

$$Re\left\{\frac{-\sum_{k=n+1}^{\infty}[1+\delta(k-1)](k-1)\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m\frac{(a)_{k-1}}{(c)_{k-1}}a_kz^{k-1}}{\sum_{k=n+1}^{\infty}[1+\delta(k-1)]\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m\frac{(a)_{k-1}}{(c)_{k-1}}a_kz^{k-1}}\right\}>-\beta|\gamma|.\tag{8}$$

Setting $z=r \quad (0 \le r < 1)$ in (8) we observe that the expression in the denominator on the left hand side of (8) is positive for $r=0$ and also for all $r \in (0,1)$. Thus by letting $r \to 1^{-}$ ،through real values (8) leads us to the desired condition (7) of the theorem.

Conversely, by applying the hypothesis (8) and setting $|z|=1,$ we find by using (7)that

$$\left|\frac{zv'}{v}-1\right| = \left| \frac{-\sum_{k=n+1}^{\infty}[1+\delta(k-1)](k-1)\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^{m}\frac{(a)_{k-1}}{(c)_{k-1}}a_k z^{k}}{z-\sum_{k=n+1}^{\infty}[1+\delta(k-1)]\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^{m}\frac{(a)_{k-1}}{(c)_{k-1}}a_k z^{k}} \right|$$

$$\le \frac{\sum_{k=n+1}^{\infty}[1+\delta(k-1)](k-1)\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^{m}\frac{(a)_{k-1}}{(c)_{k-1}}a_k z^{k-1}}{1-\sum_{k=n+1}^{\infty}[1+\delta(k-1)]\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^{m}\frac{(a)_{k-1}}{(c)_{k-1}}a_k z^{k-1}}$$

$$\le \frac{\beta|\gamma|\left(1-\sum_{k=n+1}^{\infty}[1+\delta(k-1)]\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^{m}\frac{(a)_{k-1}}{(c)_{k-1}}a_k\right)}{1-\sum_{k=n+1}^{\infty}[1+\delta(k-1)]\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^{m}\frac{(a)_{k-1}}{(c)_{k-1}}a_k}$$

$$=\beta|\gamma|.$$

Hence, by the maximum modulus principle, we have $f \in TS_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c).$

Corollary 2.1 *Let the function* f *be defined by (3)and* $f \in TR_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$, *then*

$$a_k \le \frac{\beta|\gamma|}{[1+\delta(k-1)](k+\beta|\gamma|-1)\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m \frac{(a)_{k-1}}{(c)_{k-1}}\right|}, \quad (k \ge n+1), \tag{9}$$

with equality only for functions of the form

$$f_z = z - \frac{\beta|\gamma|}{[1+\delta(k-1)](k+\beta|\gamma|-1)\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m \frac{(a)_{k-1}}{(c)_{k-1}}\right|} z^k, \quad (k \ge n+1).$$

By using the same arguments as in the proof of Theorem 2.1 we can establish the next theorem.

Theorem 2.2 *Let the function f be defined by (3). Then f belongs to the subclass* $TR_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$ *if and only if*

$$\sum_{k=n+1}^{\infty} k[1+\delta(k-1)]\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m \frac{(a)_{k-1}}{(c)_{k-1}}\right| a_k \le \beta|\gamma|, \tag{10}$$

$(z \in U, \quad 0 \le \delta \le 1, \quad 0 < \beta \le 1, \quad \gamma \in C/\{0\})$.

Corollary 2.2 *Let function* f *be defined by (3) and* $f \in TR_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$. *Then*

$$a_k \le \frac{\beta|\gamma|}{k[1+\delta(k-1)]\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m \frac{(a)_{k-1}}{(c)_{k-1}}\right|} \quad (k \ge n+1), \tag{11}$$

with equality only for functions of the form

$$f_z = z - \frac{\beta|\gamma|}{[1+\delta(k-1)]\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m \frac{(a)_{k-1}}{(c)_{k-1}}\right|} z^k, \quad (k \geq n+1).$$

3. Radius of starlikeness and convexity

We concentrate upon getting the radius of close-to-convexity, starlikeness and convexity.

Theorem 3.1 *Let the function* f *be defined by (3), belongs to the subclass* $f \in TS^{m,\delta}_{\beta,\gamma}(\lambda,\alpha,a,c)$, *then* f *is close-to-convex of order* μ, $(0 \leq \mu < 1)$ *in the disc* $|z| < r_1$, *where*

$$r_1 := \inf_{k \geq n+1}\left(\frac{(1-\mu)[1+\delta(k-1)](k+\beta|\gamma|-1)}{k\beta|\gamma|}\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m \frac{(a)_{k-1}}{(c)_{k-1}}\right|\right)^{\frac{1}{k-1}}. \tag{12}$$

Proof: It is suffices to prove that $|f'(z)-1| < 1-\mu$, $(0 \leq \mu < 1)$ for $z \in U$ such that $|z| < r_1$,

$$|f'(z)-1| = \left|\sum_{k=n+1}^{\infty} k a_k z^{k-1}\right| \leq \sum_{k=n+1}^{\infty} k a_k |z|^{k-1}.$$

Thus $|f'(z)-1| \leq 1-\mu$, if

$$\sum_{k=n+1}^{\infty} \frac{k}{1-\mu} a_k |z|^{k-1} \leq 1.$$

By making use of (7) we obtain

$$\sum_{k=n+1}^{\infty} \frac{[1+\delta(k-1)](k+\beta|\gamma|-1)}{k\beta|\gamma|}\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m \frac{(a)_{k-1}}{(c)_{k-1}}\right| \leq 1.$$

Then the inequality (13) will be true if

$$\frac{k}{1-\mu} a_k |z|^{k-1} \leq \frac{[1+\delta(k-1)](k+\beta|\gamma|-1)}{k\beta|\gamma|}\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m \frac{(a)_{k-1}}{(c)_{k-1}}\right|.$$

Thus

$$|z|:=\inf_{k\geq n+1}\left(\frac{(1-\mu)[1+\delta(k-1)](k+\beta|\gamma|-1)}{k\beta|\gamma|}\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m\frac{(a)_{k-1}}{(c)_{k-1}}\right|\right)^{\frac{1}{k+1}}.$$

This completes the proof.

Theorem 3.2

Let the function f be defined by (3) belongs to the subclass $f\in TS_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$. Then f is starlike of order μ, $(0\leq\mu<1)$ in the disc $|z|<r_2$, that is,

$$r_2:=\inf_{k\geq n+1}\left(\frac{(1-\mu)[1+\delta(k-1)](k+\beta|\gamma|-1)}{(k-\mu)\beta|\gamma|}\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m\frac{(a)_{k-1}}{(c)_{k-1}}\right|\right)^{\frac{1}{k-1}}.$$

Proof:

It is suffices to show that $\left|\frac{zf'(z)}{f(z)}-1\right|<1-\mu,\ (0\leq\mu<1)$ for $(z\in U)$ such that $|z|<r_2$, we have

$$\left|\frac{zf'(z)}{f(z)}-1\right|=\left|\frac{\sum_{k=n+1}^{\infty}(k-1)a_kz^{k-1}}{1-\sum_{k=n+1}^{\infty}a_kz^{k-1}}\right|\leq\frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=n+1}^{\infty}a_k|z|^{k-1}}.$$

Thus $\left|\frac{zf'(z)}{f(z)}-1\right|\leq1-\mu$, if

$$\sum_{k=n+1}^{\infty}\frac{k-\mu}{1-\mu}a_k|z|^{k-1}\leq1.$$

In virtue of (7) we have

$$\sum_{k=n+1}^{\infty}\frac{[1+\delta(k-1)](k+\beta|\gamma|-1)}{k\beta|\gamma|}\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m\frac{(a)_{k-1}}{(c)_{k-1}}\right|\leq 1.$$

Then the inequality (14) will be true if

$$\frac{k-\mu}{1-\mu}a_k\,|z|^{k-1}\leq\frac{[1+\delta(k-1)](k+\beta|\gamma|-1)}{k\beta|\gamma|}\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m\frac{(a)_{k-1}}{(c)_{k-1}}\right|.$$

Thus

$$|z|:=\inf_{k\geq n+1}\left(\frac{(1-\mu)[1+\delta(k-1)](k+\beta|\gamma|-1)}{(k-\mu)\beta|\gamma|}\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m\frac{(a)_{k-1}}{(c)_{k-1}}\right|\right)^{\frac{1}{k+1}}.$$

This completes the proof.

Corollary 3.1 *Let the function* f *be defined by (3) belongs to the subclass* $f\in TS_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$. *Then* f *is convex of order* μ, $(0\leq\mu<1)$ *in the disc* $|z|<r_3$, *that is,*

$$r_3:=\inf_{k\geq n+1}\left(\frac{(1-\mu)[1+\delta(k-1)](k+\beta|\gamma|-1)}{k(k-\mu)\beta|\gamma|}\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m\frac{(a)_{k-1}}{(c)_{k-1}}\right|\right)^{\frac{1}{k-1}}.$$

By using the same arguments as in the proofs of Theorems 3.1 and 3.2 we can obtain the radii of close-to-convexity, starlikeness and convexity for the subclass $f\in TR_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$.

Theorem 3.3 *Let the function* f *be defined by (3), belongs to the subclass* $f\in TR_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$, *then* f *is close-to-convex of order* μ, $(0\leq\mu<1)$ *in the disc* $|z|<\rho_1$, *where*

$$\rho_1:=\inf_{k\geq n+1}\left(\frac{(1-\mu)[1+\delta(k-1)]}{\beta|\gamma|}\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m\frac{(a)_{k-1}}{(c)_{k-1}}\right|\right)^{\frac{1}{k-1}}.$$

Theorem 3.4 *Let the function* f *be defined by (3) belongs to the subclass* $f \in TR_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$. *Then* f *is starlike of order* μ, $(0 \le \mu < 1)$ *in the disc* $|z| < \rho_2$, *that is,*

$$\rho_2 := \inf_{k \ge n+1} \left(\frac{k(1-\mu)[1+\delta(k-1)](k+\beta|}{(k-\mu)\beta|\gamma|} \left| \left(\frac{\lambda(k-1)+1+l}{1+l} \right)^m \frac{(a)_{k-1}}{(c)_{k-1}} \right| \right)^{\frac{1}{k-1}}.$$

Corollary 3.2 *Let the function* f *be defined by (3) belongs to the subclass* $f \in TR_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$. *Then* f *is convex of order* μ, $(0 \le \mu < 1)$ *in the disc* $|z| < \rho_3$, *that is,*

$$\rho_3 := \inf_{k \ge n+1} \left(k \, \frac{(1-\mu)[1+\delta(k-1)]}{(k-\mu)\beta|\gamma|} \left| \left(\frac{\lambda(k-1)+1+l}{1+l} \right)^m \frac{(a)_{k-1}}{(c)_{k-1}} \right| \right)^{\frac{1}{k-1}}.$$

4. Closure Theorems

Let the function $f_j \, (j = 1,2,3...,p)$ defined by

$$f_j(z) = z - \sum_{k=n+1}^{\infty} a_{k,j} z^k, (z \in \mathrm{U}).$$

We obtain the following results for the closure of functions in the subclasses $TS_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$ and $TR_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$.

Theorem 4.1 *Let the function* $f_j \, (j = 1,2,3....,p)$ *be defined by (16) belongs to the subclass* $f \in TR_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$, *and let* $c_j \ge 0, (j = 1,2,3....,p)$ *such that* $\sum_{J=1}^{p} c_j = 1$. *Then the function* h *defined by*

$$h = \sum_{J=1}^{p} c_j f_j,$$

is also in the subclass $TS_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$.

Proof: In virtue of the definition of h, we can write

$$h=\sum_{J=1}^{p}c_j\,(z-\sum_{k=n+1}^{\infty}a_{k,j}z^k)$$

$$=(\sum_{J=1}^{p}c_j)z-\sum_{k=n+1}^{\infty}(\sum_{J=1}^{p}c_ja_{k,j})z^k$$

$$=z-\sum_{k=n+1}^{\infty}(\sum_{J=1}^{p}c_ja_{k,j})z^k.$$

Since the functions f_j are in $TS_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$, for every $j=1,2,3,...,p$, we have

$$\sum_{k=n+1}^{\infty}[1+\delta(k-1)](k+\beta|\gamma|-1)\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m\frac{(a)_{k-1}}{(c)_{k-1}}\right|a_{k,j}\le\beta|\gamma|.$$

Hence we get

$$\sum_{k=n+1}^{\infty}[1+\delta(k-1)](k+\beta|\gamma|-1)\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m\frac{(a)_{k-1}}{(c)_{k-1}}\right|(\sum_{J=1}^{p}c_ja_{k,j})\le\beta|\gamma|.$$

$$\sum_{J=1}^{p}c_j\sum_{k=n+1}^{\infty}[1+\delta(k-1)](k+\beta|\gamma|-1)\left|\left(\frac{\lambda(k-1)+1+l}{1+l}\right)^m\frac{(a)_{k-1}}{(c)_{k-1}}\right|a_{k,j}$$

$$\le\sum_{J=1}^{p}c_j\,(\beta|\gamma|)=\beta|\gamma|,$$

which implies that h is in the subclass $TS_{\beta,\gamma}^{m,\delta}(\lambda,l,a,c)$. Thus, the proof of the theorem is complete.

Many other work on analytic functions related to derivative operator and integral operator can be read in [2],[4] and [5] .

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