	مجـلـة الـتربـوي Journal of Educational ISSN: 2011- 421X Arcif Q3	معامل التأثير العربي 1.5 العدد 19

مجلة التربوي

مجلة علمية محكمة تصدر عن كلية التربية

جامعة المرقب

العدد التاسع عشر

يوليو 2021م

هيئـــة تحرير

مجلة التربوي

- المجلة ترحب بما يرد عليها من أبحاث وعلى استعداد لنشرها بعد التحكيم .
- المجلة تحترم كل الاحترام آراء المحكمين وتعمل بمقتضاها .
- كافة الآراء والأفكار المنشورة تعبر عن آراء أصحابها ولا تتحمل المجلة تبعاتها .
- يتحمل الباحث مسؤولية الأمانة العلمية وهو المسؤول عما ينشر له .
- البحوث المقدمة للنشر لا ترد لأصحابها نشرت أو لم تنشر .

ضوابط النشر:

يشترط في البحوث العلمية المقدمة للنشر أن يراعى فيها ما يأتي :

- أصول البحث العلمي وقواعده .
- ألا تكون المادة العلمية قد سبق نشرها أو كانت جزءا من رسالة علمية .
- يرفق بالبحث تزكية لغوية وفق أنموذج معد .
- تعدل البحوث المقبولة وتصحح وفق ما يراه المحكمون .
- التزام الباحث بالضوابط التي وضعتها المجلة من عدد الصفحات ، ونوع الخط ورقمه ، والفترات الزمنية الممنوحة للتعديل ، وما يستجد من ضوابط تضعها المجلة مستقبلا .

تنبيهات :

- للمجلة الحق في تعديل البحث أو طلب تعديله أو رفضه .
- يخضع البحث في النشر لأولويات المجلة وسياستها .
- البحوث المنشورة تعبر عن وجهة نظر أصحابها ، ولا تعبر عن وجهة نظر المجلة .

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Least-Squares Line

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الملخص

في هذه الورقة سندرس تطبيق عملي لفضاء الضرب لمحاولة لحل نظام خطي في صورة AX=b عندما n>m من المتغيرات (m &n) من المعادلات

Introduction:

We derive the method of finding a polynomial that best fits given data points a method that is extremely important to the natural sciences , social sciences , and engineering .
Let x be the vector of variables and b be the vector of constants.
We have seen that a system $Ax = b$ of n equations in n variables, where A is invertible, has the unique solution $x = A^{-1}b$.
However , if $Ax = b$ is a system of n equations in m variables , where n $\rangle$ m , the system does not , in general , have a solution and it is then said to be overdetermined .
A is not a square matrix for such a system, and A^{-1} does not exist. We will introduce a matrix called the pseudoinverse of A, denoted pinv (A), that leads to a least-squares solution x = pinv (A) b for an overdetermined system. This is not a true solution, but it is in some sense the closest we can get to a true solution for the system. We will see an application of overdetermined in finding curves that "best" fit data.

Definition [3]
Let A $\in$ Mm×n . We define the adjoint of A to be the n×m matrix adj(A) or A^* such that $(A^*)_{ij} = A_{ji}$ for all i, j .

Definition [3]
The transpose A^t of an n×m matrix A is the n×m obtained from A by interchanging the rows with the columns; that is $(A^t)_{ij} = A_{ji}$

Definition :[11]
Let A be a matrix. The matrix $(A^t A)^{-1} A^t$ is called the pseudoinverse of A, and is denoted pinv (A).
We have seen that not every matrix has an inverse. Similarly, not every matrix has a pseudoinverse. The matrix A has pseudoinverse if $(A^t A)^{-1}$ exists.
Example (1):-

Find the pseudoinverse of $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 4 \end{bmatrix}$

Solution:

We compute the pseudoinverse of A in stages

$$A^t A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 7 & 29 \end{bmatrix}$$

$$(A^t A)^{-1} = \frac{1}{|A^t A|} \operatorname{adj}(A^t A) = \frac{1}{125} \begin{bmatrix} 29 & -7 \\ -7 & 6 \end{bmatrix}$$

$$\operatorname{pinv}(A) = (A^t A)^{-1} A^t = \frac{1}{125} \begin{bmatrix} 29 & -7 \\ -7 & 6 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$

$$= \frac{1}{25} \begin{bmatrix} 3 & -10 & 6 \\ 1 & 5 & 2 \end{bmatrix}$$

Now we will use the concept of pseudoinverse to further our understanding of systems of linear equations.

Let $Ax = b$ be a system of n linear equation in m variables with $n \rangle m$, where A is of rank m . Multiply each side of this matrix equation by A^t, to get:-

$A^t Ax = A^t b$ normal equation

such that $A^t A$ is symmetric matrix and Least – Squares solution x satisfies :

$A^t (b - Ax) = 0$

The matrix $A^t A$ can be shown to be invertible for such system.

Multiply each side of this equation by $(A^t A)^{-1}$ and solve for x to get

$$(A^t A)^{-1} (A^t Ax) = (A^t A)^{-1} A^t b$$

$$[(A^t A)^{-1} (A^t A)]x = (A^t A)^{-1} A^t b$$

$$x = (A^t A)^{-1} A^t b$$

$$= pinv\ (A) b$$

This value of x is called the least-squares Solution to the system of equations.

Result:-

$$Ax = b \qquad\qquad x = pinv\,(A) b$$

System Least-squares solution

Let $Ax = b$ be a system of n linear equations in m variables with $n \rangle m$, where A is of rank m. This system has a least-squares solution. If the system has a unique solution, the least-squares solution is that unique solution. If the system is over determined, the least squares solution is the closest we can get to a true solution. The system cannot have many solutions.

Example (2):-
Find the least-squares solution of the following overdetermined system of equations and sketch the solution

$$x + y = 6$$
$$-x + y = 3$$
$$2x + 3y = 9$$

Solution:-
The matrix of coefficients is

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 3 \end{bmatrix} \quad and \quad b = \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix}$$

The column vectors of A are linearly independent. Thus the rank of A is 2.
This system has a least-squares solution.
We compute pinv (A).

$$A^t = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

$$A^t A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 11 \end{bmatrix}$$

$$(A^t A)^{-1} = \frac{1}{|A^t A|} adj\ (A^t A) = \frac{1}{30} \begin{bmatrix} 11 & -6 \\ -6 & 6 \end{bmatrix}$$

$$pinv\,(A) = (A^t A)^{-1} A^t = \frac{1}{30} \begin{bmatrix} 11 & -6 \\ -6 & 6 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

$$= \frac{1}{30} \begin{bmatrix} 5 & -17 & 4 \\ 0 & 12 & 6 \end{bmatrix}$$

The least-squares solution is

$$pinv(A)b = \frac{1}{30}\begin{bmatrix} 5 & -17 & 4 \\ 0 & 12 & 6 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \\ 3 \end{bmatrix}$$

The least-squares solution is the point $P(\frac{1}{2}, 3)$ in figure 1

Now we will see how Least-squares solutions can be used to find curves that best fit given data.

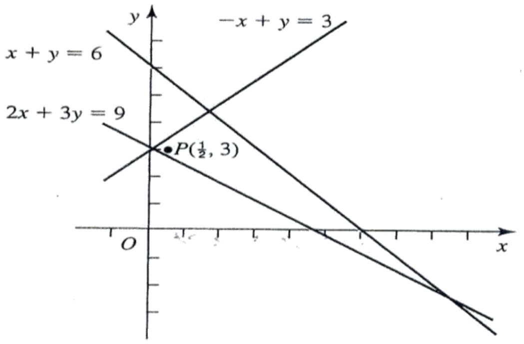

Figure 1

Least-squares Curves

Many branches of science and business use equations based on data that has been determined from experimental results.
In many applications, however, there is too much data to lead to an equation that exactly fits all data. One then uses the equation of a line or curve that in some sense "best" fits all the data. For example, suppose the data consists of the points $(x_1, y_1), \ldots, (x_n, y_n)$, shown in figure 2 (a). These points lie approximately on a line. We would want the equation of the line that best fits these points. On the other hand, the points might closely fit a parabola, as shown in figure 2(b). We would then want to find the parabola that most closely fits these points.
Many criteria can be used for the best fit in such cases. The one that has generally been found to be most satisfactory is called the least-squares line or curve – found by solving an overdetermined system of equations. The least-squares line and curve is such that $d_1^2 + \ldots + d_n^2$ in figure 2 is a minimum.
We want the best fit to discrete set of data points over a given interval.

We illustrate how to fit a least-squares polynomial to given data. The method involves constructing a system of linear equations. The least-squares solution to this system of equations gives the coefficients of the polynomial.

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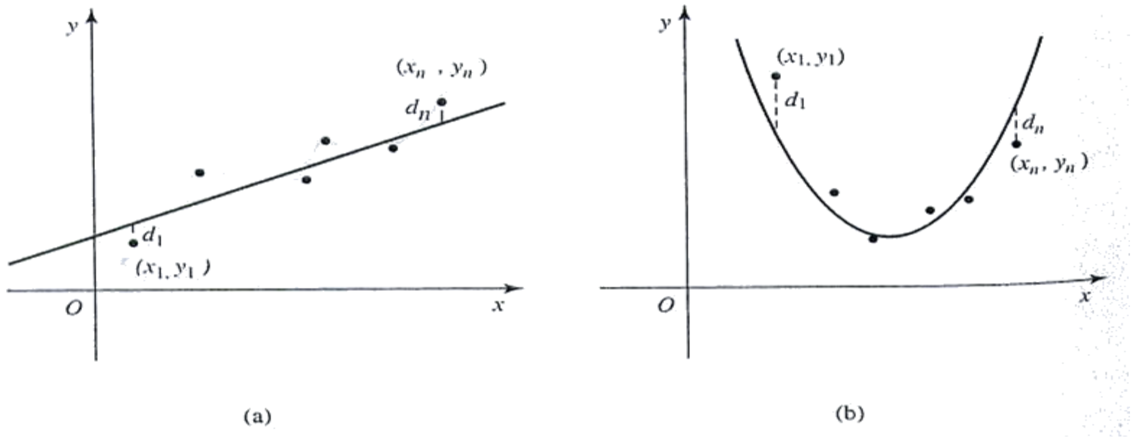

Figure 2

Example (3):-
Find the least squares line for the following data points (1,1),(2,2.4),(3,3.6)(4,4) .
Solution:-
Let the equation of the line be $y = a_1 + a_a x$ Substituting for these points into equation of the line, we get the overdetermined system.

$$\begin{aligned} a_1 + a_2 &= 1 \\ a_1 + 2a_2 &= 2.4 \\ a_1 + 3a_2 &= 3.6 \\ a_1 + 4a_2 &= 4 \end{aligned}$$

We find the least squares solution.
The matrix of coefficients A and column vector d are as follows:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \quad and \quad d = \begin{bmatrix} 1 \\ 2.4 \\ 3.6 \\ 4 \end{bmatrix}$$

It can be shown that

$$pinv(A) = (A'A)^{-1}A^t = \frac{1}{20}\begin{bmatrix} 20 & 10 & 0 & -10 \\ -6 & -2 & 2 & 6 \end{bmatrix}$$

The least squares solution is

$$\left[(A^tA)^{-1}A^t\right]d = \frac{1}{20}\begin{bmatrix} 20 & 10 & 0 & -10 \\ -6 & -2 & 2 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2.4 \\ 3.6 \\ 4 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 1.02 \end{bmatrix}$$

$$thus \quad a_1 = 0.2 \qquad a_2 = 1.02$$

The equation of the least-squares line for this data is
y = 0.2 +1.02x
This is the line that is generally considered to be the line of best fit for these points. See Figure 3

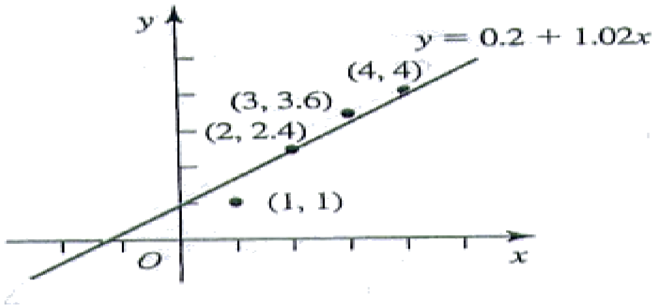

Figure 3

Example (4):-
Find the least-squares parabola for the following data points
(1, 7),(2, 2),(3, 1),(4, 3)
Solution:-
Let the equation of the parabola be

$$y = a_1 + a_2 x + a_3 x^2$$

Substituting for these points into the equation of the parabola, we get the system

$$a_1 + a_2 + a_3 = 7$$
$$a_1 + 2a_2 + 4a_3 = 2$$
$$a_1 + 3a_2 + 9a_3 = 1$$
$$a_1 + 4a_2 + 16a_3 = 3$$

We find the least-squares solution.
The matrix of coefficients A and column vector d are as follows

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \quad and \quad d = \begin{bmatrix} 7 \\ 2 \\ 1 \\ 3 \end{bmatrix}$$

Then

$$pinv\,(A) = (A^t A)^{-1} A^t = \frac{1}{20} \begin{bmatrix} 45 & -15 & -25 & 15 \\ -31 & 23 & 27 & -19 \\ 5 & -5 & -5 & 5 \end{bmatrix}$$

The least-squares solution is

$$\left[(A^tA)^{-1}A^t\right]d=\frac{1}{20}\begin{bmatrix}45 & -15 & -25 & 15\\ -31 & 23 & 27 & -19\\ 5 & -5 & -5 & 5\end{bmatrix}\begin{bmatrix}7\\2\\1\\3\end{bmatrix}=\begin{bmatrix}15.25\\-10.05\\1.75\end{bmatrix}$$

Then

$$a_1 = 15.25 \qquad a_2 = 10.05 \qquad a_3 = 1.75$$

The equation of the least-squares parabola for these data points is

$y = 15.25 - 10.05x + 1.75x^2$,

as shown in figure 4

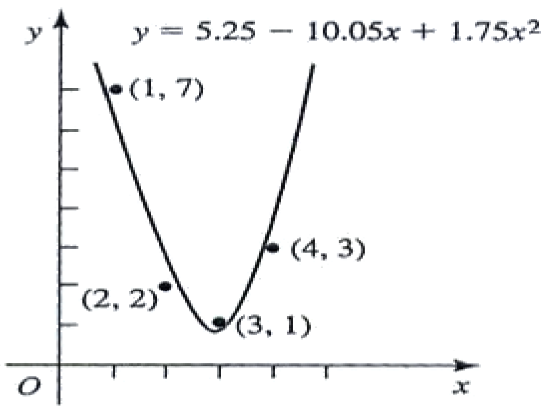

Figure 4

THEOREM 1: [11]

Let $(x_1, y_1), \ldots, (x_n, y_n)$ be a set of n data points. Let $y = a + a_1x + \ldots + a_mx^m$ be a polynomial of a degree $m(n\rangle m)$ that is to be fitted to these points. Substituting these points into the polynomial leads to a system $Ax = b$ of n linear equations in the m+1 variables $a_0, \ldots, a_m$ where

$$A=\begin{pmatrix}1 & x_1\ldots & x_1^m\\ \vdots & \vdots & \vdots\\ 1 & x_n\cdots & x_n^m\end{pmatrix} \quad \textit{and} \quad b=\begin{bmatrix}y_1\\ \vdots\\ y_n\end{bmatrix}$$

The least-squares solution to this system gives the coefficients of the least-squares polynomial for these data points.

Example (5):-

(Hooke's Law) states that when a force is applied to a spring, the length of the spring will be a linear function of the force. If L is the length of the spring when the force is F, this means that there exist (spring) constants a and b such that

$$L = a_1 + a_2F$$

We shall now see that the spring constants, and hence the relationship between length of the spring L and force F can be found by using the method of least squares . Let various weights be suspended from the spring, and the length of the spring measured in each case.

Let the results be as following:-

Force, F (in ounces)	2	4	6	8
Length, L (in inches)	8.2	11.6	14.3	17.5

Write these statistics as points, where the first component is F and the second component is L. We get

(2 , 8.2) , (4 , 11.6) , (6 , 14.3) , (8 , 17. 5)

In theory, these points should all lie on a straight line.

The least – square line through these points will give the most satifactory equation for the line. We get the system:

$$\begin{aligned} a_1 + 2a_2 &= 8.2 \\ a_1 + 4a_2 &= 11.6 \\ a_1 + 6a_2 &= 14.3 \\ a_1 + 8a_2 &= 17.5 \end{aligned}$$

The matrix of coefficient A and constant column matrix d are as follows.

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 6 \\ 1 & 8 \end{bmatrix} \quad \text{and} \quad d = \begin{bmatrix} 8.2 \\ 11.6 \\ 14.3 \\ 17.5 \end{bmatrix}$$

We get

$$\text{pinv}\,(A) = (A^t A)^{-1} A^t = \begin{bmatrix} 1 & 0.5 & 0 & -0.05 \\ -0.15 & -0.05 & 0.05 & 0.15 \end{bmatrix}$$

The least – squares solution is:

$$\left[(A^tA)^{-1}A^t\right]d = \begin{bmatrix} 1 & 0.5 & 0 & -0.05 \\ -0.15 & -0.05 & 0.05 & 0.15 \end{bmatrix} \begin{bmatrix} 8.2 \\ 11.6 \\ 14.3 \\ 17.5 \end{bmatrix} = \begin{bmatrix} 5.25 \\ 1.53 \end{bmatrix}$$

Thus the spring constants are a_1 = 5.25 and a_2= 1.53, the equation for the spring is L = 5.25 + 1.53 F.

Thus, for example, when a weight of 20 ounces is attached to the spring, we can expect the length of the spring to be approximately: 5.25 + (1.53 x 20); that is 35.85 inches.

THEOREM 2 [11]

Let W be the subspace of R^n generated by linearly independent vectors $u_1,\ldots,u_m$.

Let $A=[u_1,\ldots,u_m]$. be the matrix where columns are the vectors $u_1, u_2,\ldots,u_m$. The projection of a vector y onto W is given by

$$proj_w y = A \quad pinv(A) y$$

A pinv(A) is called a projection matrix .

Example (6):-
Find the projection matrix for the plane $x - 2y - z = 0$ in R^3.
Use this matrix to find the projection of the vector (1 , 2 , 3) onto this plane.
Solution:
Let W be the subspace of vectors that lie in this plane. W consists of vectors of the form (x, y, z) where x = 2y + z.
Thus W = {(2y + z , y , z)}. We can write
W= {(y(2,1,0)+z(1,0,1)}.
Therefore, W is the space generated by the vectors (2, 1, 0) and (1,0,1,) . Let A be the matrix having these vectors as columns.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

It can be shown that:

$$pinv(A) = \frac{1}{6}\begin{bmatrix} 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix}$$

The projection matrix is

$$A \, pinv(A) = \frac{1}{6}\begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix}$$

The projection of (1 , 2 , 3) onto W is computed by multiplying this vector, in column from, by A pinv (A) We get

$$\frac{1}{6}\begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

Thus the projection of (1, 2, 3) onto plane x -2y – z = 0
is (2 ,0 , 2) .

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