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مجلة التربوي
مجلة علمية محكمة تصدر عن كلية التربية
جامعة المرقب

العدد التاسع عشر
يوليو 2021م

هيئـــة تحرير
مجلة التربوي

- المجلة ترحب بما يرد عليها من أبحاث وعلى استعداد لنشرها بعد التحكيم .
- المجلة تحترم كل الاحترام آراء المحكمين وتعمل بمقتضاها .
- كافة الآراء والأفكار المنشورة تعبر عن آراء أصحابها ولا تتحمل المجلة تبعاتها .
- يتحمل الباحث مسؤولية الأمانة العلمية وهو المسؤول عما ينشر له .
- البحوث المقدمة للنشر لا ترد لأصحابها نشرت أو لم تنشر .

ضوابط النشر:

يشترط في البحوث العلمية المقدمة للنشر أن يراعى فيها ما يأتي :

- أصول البحث العلمي وقواعده .
- ألا تكون المادة العلمية قد سبق نشرها أو كانت جزءا من رسالة علمية .
- يرفق بالبحث تزكية لغوية وفق أنموذج معد .
- تعدل البحوث المقبولة وتصحح وفق ما يراه المحكمون .
- التزام الباحث بالضوابط التي وضعتها المجلة من عدد الصفحات ، ونوع الخط ورقمه ، والفترات الزمنية الممنوحة للتعديل ، وما يستجد من ضوابط تضعها المجلة مستقبلا .

تنبيهات :

- للمجلة الحق في تعديل البحث أو طلب تعديله أو رفضه .
- يخضع البحث في النشر لأولويات المجلة وسياستها .
- البحوث المنشورة تعبر عن وجهة نظر أصحابها ، ولا تعبر عن وجهة نظر المجلة .

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Totally Semi-open Functions in Topological Spaces

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Abstract

In this paper, we introduce the concept of totally semi-open functions in topological spaces. Some interesting results and properties of totally semi-open functions are investigated and proven.

Keywords: *Semi-open function, Semi-open set, Semi-closed set, Semi-clopen set, Totally open function.*

1 Introduction

The notion of semi-open sets and semi-continuity was first introduced and investigated by Levine [8] in 1963. In 1969, Biswas [2] defined and studied semi-open functions. Irresolute functions and semi-homeomorphisms were introduced and studied by Crossley and Hildebrand [4]. Nour [11] defined totally semi-continuous and strongly semi-continuous functions. Benchalli and Neeli in [1] introduced and studied semi-totally continuous and semi-totally open functions. Garg and Shivaraj in [7] introduced the concept of preclosed mapping.

In this paper, a new generalization of semi-open functions is introduced and studied. We define totally semi-open functions and prove some interesting results and properties in this connection.

2 Preliminaries

Throughout this paper, X will always denote a topological space. If A is a subset of X, then $\bar{A}$ and A° respectively denote the closure and the interior of A in X.

Definition 1. A subset A of a topological space (X, τ) is said to be

1) semi-open set [8] if there exists an open set U in X such that $U \subseteq A \,\square\, \subseteq \overline{U}$, i.e., $A \subseteq \overline{A^{\circ}}$.

2) semi-closed set [4] if A is the complement of a semi-open set, i.e., $(\bar{A})^{\circ} \subseteq A$. **Definition 2.** [4] (1) The semi-closure of a set A in X is the intersection of all semi-closed sets that contains A; this set is denoted by $\underline{A}$.

2) The semi-interior of a set A in X is the union of all semi-open sets of X contained in A; this set is denoted by $A_{\circ}$

3) A point $x \in X$ is said to be a semi-limit point of a set A in X if every semi-open set containing x contains at least one point of A different from x itself.

Theorem 1. [8] If $\{A_i\}_{i\in I}$ a collection of semi-open sets in a topological space X then $\bigcup_{i\in I} A_i$ is semi-open.

Definition 3. A function $f : X \to Y$ is said to be

1) semi-continuous [8] if the inverse image of each open subset of Y is semi-open in X.

2) semi-open [2] if $f(U)$ is semi- open in Y for each open set U in X .

3) totally semi-continuous [11] if the inverse image of every open subset of Y is semi- clopen in X .

4) irresolute [4] if the inverse image of every semi- open set in Y is semi-open in X.

5) pre semi-open [4] if the image of every semi-open set in X is semi-open in Y.

6) semi-totally continuous [1] if the inverse image of every semi- open subset of Y is clopen in X.

7) semi-totally open [1] if the image of every semi-open set in X is clopen in Y.

Definition 4. [9] A topological space X is said to be

1) semi-T_0 if for each pair of distinct points in X, there exists a semi-open set containing one point but not the other.

2) semi-T_1 if for each pair of distinct points x and y of X, there exist semi-open sets U and V such that $x \in U, y \notin U$ and $x \notin V$, $y \in V$.

3) semi-T_2 if for each pair of distinct points x and y of X, there exist semi-open sets U and V such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Definition 5. [10**]** A topological space X is said to be semi-connected if X is not the union of two nonempty disjoint semi-open subsets of X.

Definition 6. [5**]** A topological space X is said to be semi-compact if any semi-open cover of X has a finite subcover.

3 Totally Semi-open Functions

In this section, we introduce the concept of totally semi-open functions. Further, we study many properties and prove some results.

Definition 7. A function $f : X \rightarrow Y$ is said to be totally semi-open if the image of every open set in X is semi-clopen in Y.

Example 1. Let $X = \{1,2,3\}$, $\tau = \{\phi, X, \{1\}\}$ and $\tau' = \{\phi, X, \{2\}, \{1,3\}\}$. The function $g: (X, \tau) \rightarrow (X, \tau')$ defined by $g(1) = 2$, $g(2) = 1$, $g(3) = 3$ is totally semi-open.

Example 2. Let $I_{\mathbb{R}}: \mathbb{R} \rightarrow \mathbb{R}$ be the identity function. If $A = (-\infty, 0) \cup (0, \infty)$, then A is open, but $f(A) = A$ is not semi-clopen in $\mathbb{R}$. So, $I_{\mathbb{R}}$ is not totally semi-open.

Theorem 2. Let $f : X \rightarrow Y$ be a totally semi-open injective function and let $A \subseteq X$. For any $x \in X$, if $f(x)$ is a semi-limit point of $f(A)$ then x is a limit point of A.

Proof. Suppose that x is not a limit point of A, so there exists an open set U in X such that $x \in U$ and $U \cap A \backslash \{x\} = \phi$. But then $f(U) \cap f(A) \backslash \{f(x)\} = \phi$ and hence $f(x)$ is not a semi-limit point of $f(A)$ since $f(U)$ is semi-clopen in Y.

Definition 8. A function $f : X \rightarrow Y$ is said to be totally semi-closed if the image of every closed set in X is semi-clopen in Y.

Lemma 1. A bijection $f: X \rightarrow Y$ is totally semi-open if and only if it is totally semi-closed.

Proof. Suppose f is totally semi-open and let $A \subseteq X$ be a closed set, then $f(X \backslash A) = Y \backslash f(A)$ is semi-clopen in Y if and only if $f(A)$ is also semi-clopen in Y.

Theorem 3. If $f : X \rightarrow Y$ is a totally semi-open function then for all $A \subseteq X$:

1) $f(A^{\circ}) \subseteq (f(A))_{\circ}$

2) $\underline{f(A)} \subseteq f(\bar{A})$

Proof. 1) Since $A^{\circ} \subseteq A$, then $f(A^{\circ}) \subseteq f(A)$. Hence, $f(A^{\circ}) = (f(A^{\circ}))_{\circ} \subseteq (f(A))_{\circ}$, since $f(A^{\circ})$ is semi-open in Y.

2) Since $A \subseteq \bar{A}$, then $f(A) \subseteq f(\bar{A})$. From Lemma 1 we have f is also totally semi-closed, so $f(\bar{A})$ is semi-clopen in Y. Hence, $\underline{f(A)} \subseteq \underline{f(\bar{A})} = f(\bar{A})$.

Lemma 2. [8] Let $\{X_i\}_{i \in \mathbb{N}}$ be a collection of topological spaces and for each i, A_i is a semi-open (resp. semi-closed) set in X_i, then $A = \prod_{i \in I} A_i$ is semi-open (resp. semi-closed) in the product space $\prod_{i \in I} X_i$.

Theorem4. Let $\{f_i: X \to X_i\}_{i\in\mathbb{N}}$ be a collection of totally semi-open functions. Then the function $f : X \longrightarrow \prod_{i\in\mathbb{N}} X_i$ defined by $f(x) = (f_i(x))_{i\in I}$ is totally semi-open.

Proof. Let U be an open set in X, then $f(U) = \prod_{i\in\mathbb{N}} f_i(U)$, where $f_i(U)$ is semi-clopen in X for all $i \in \mathbb{N}$, so the proof follows directly from Lemma 2.

Theorem 5. Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function of f where $g(x) = (x, f(x))$ for each $x \in X$. If g is totally semi-open then f is also totally semi-open.

Proof. Let U be an open set in X, then $g(U) = U \times f(U)$ is semi-clopen in $X \times Y$. Since $U \times f(U)$ is semi-open, $U \times f(U) \subseteq \overline{(U \times f(U))^{\circ}} = \overline{U^{\circ}} \times \overline{(f(U))^{\circ}}$, so $f(U) \subseteq \overline{(f(U))^{\circ}}$ and $f(U)$ is semi-open in Y. Similarly, we can prove that $f(U)$ is semi-closed in Y. Therefore, $f(U)$ is semi-clopen in Y and f is totally semi-open function.

Definition 9. A function $f : X \to Y$ is said to be totally open if the image of every open set in X is clopen in Y.

Theorem 6. Every totally open function is totally semi-open.

Proof. The proof follows directly from the fact that every clopen set is also semi-clopen.

Remark 1. The converse of Theorem 6 is not true as shown by the following example.

Example 3. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{b, c\}\}$ and $\tau' = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$. The function $g: (X, \tau) \to (X, \tau')$ defined by $g(x) = x$ is a totally semi-open function, but $g$is not totally open.

Theorem 7. Every totally semi-open function is semi-open.

Proof. Let $f : X \to Y$ be a totally semi-open function and $U \subseteq X$ be an open set, then $f(U)$ is semi-clopen in Y, so $f(U)$ is semi-open in Y.

Remark 2. The converse of the above theorem is false as shown by:

Example 4. Let $X = \{1,2,3\}$ and $\tau = \{\phi, X, \{1\}\}$, then the identity function on X is semi-open but not totally semi-open.

Theorem 8. Every semi-totally open function is totally semi-open.

Proof. Let $f : X \to Y$ be a semi-totally open function and $U \subseteq X$ be an open set. So, U is also semi-open and then $f(U)$ is clopen in Y. Since a clopen set is also semi-clopen, the proof is complete.

Remark 3. The converse of Theorem 8 is not true, for example, the function g defined in Example 3 is totally semi-open but not semi-totally open.

Remark 4. A totally semi-open function need not be pre-semi-open. Also, a pre-semi-open function need not be totally-semi-open as is illustrated in the following examples:

Example 5. The identity function on $\mathbb{R}$ is pre-semi-open but not totally semi-open.

Example 6. Let $X = \{1,2,3\}$, $\tau = \{\phi, X, \{2\}\}$ and $\tau' = \{\phi, X, \{2\}, \{1,3\}\}$. The function $g: (X, \tau) \to (X, \tau')$ defined by $g(x) = x$ is totally semi-open but not pre-semi-open.

Remark 5. Example 5 and Example 3 also illustrate that totally semi-open functions and open functions are independent notions.

Remark 6. The composition of two totally semi-open functions need not be a totally semi-open function as is illustrated in the following example.

Example 7. Let $X = \{1,2,3\}$, $\tau = \{\phi, X, \{2,3\}\}$ and $\tau' = \{\phi, X, \{1\}, \{3\}, \{1,3\}\}$ and let $X^* = \{1,2,3,4\}$ and $\tau^* = \{\phi, X, \{1\}, \{1,3\}, \{3\}\}$. Let $f: (X, \tau) \to (X^*, \tau^*)$ be defined as $f(1) = 2, f(2) = 1, f(3) = 4$ and let $g: (X^*, \tau^*) \to (X, \tau')$ be defined by $g(1) = g(3) = 1, g(2) = 2, g(4) = 3$, then both f and g are totally semi-open. But the composition $g \circ f$ is not a totally semi-open function because $(g \circ f)(\{2,3\}) = \{1,3\}$ and $\{1,3\}$ is not semi-clopen in Z although $\{2,3\}$ is open in X.

The proof of the following theorem is obvious.

Theorem 9. If $f : X \to Y$ is open and $g : Y \to Z$ is totally semi-open, then $g \circ f : X \to Z$ is totally semi-open.

Theorem 10. If $f : X \to Y$ and $g : Y \to Z$ are functions and $g \circ f : X \to Z$ is totally semi-open then:

(1) if f is continuous and surjective, then g is totally semi- open.

(2) if g is semi-totally continuous and injective, then f is totally semi-open.

Proof. (1) Let U be an open set in Y then $f^{-1}(U)$ is open in X since f is continuous, Since f is surjective then $(g \circ f)\left(f^{-1}(U)\right) = g(U)$. Since $g \circ f$ is totally semi-open then $g(U)$ is semi- clopen in Z.

(2) Let U be an open set in X, then $(g \circ f)(U) = g(f(U))$ is semi-clopen (and hence semi-open) in Z since $g \circ f$ is totally semi-open. But g is totally semi-continuous and injective, so $g^{-1}((g \circ f)(U)) = f(U)$ is clopen and hence semi-clopen in Y.

The proof of the following theorem is obvious and hence omitted.

Theorem 11. If $f : X \to Y$ is a totally semi-open function and A is an open subset of X, then the restriction function $f|A : A \to Y$ is also totally semi-open.

Remark 7. Let $f : X \to Y$ be a function and $X = A \cup B$ where A, B are open. If both $f|A$ and $f|B$ are totally semi-open, then f need not be totally semi-open since, from Theorem 1, the union of semi-clopen sets is not necessarily semi-clopen.

Theorem 12. If $f : X \to Y$ is a totally semi-open bijection and X is a T_0space then Y is a semi-T_2 space.

Proof. Let $x, y \in Y$ and $x \neq y$ then $f^{-1}(x) \neq f^{-1}(y)$. Since X is a T_0-space, then there exist an open set U such that $f^{-1}(x) \in U$ and $f^{-1}(y) \notin U$. Now, $f(U)$, $Y - f(U)$ are disjoint semi-clopen sets (hence semi-open sets) and $x \in f(U)$, $y \in Y - f(U)$. Therefore, Y is a semi-T_2 space.

Corollary 1. If $f : X \to Y$ is a totally semi-open bijection and X is a T_1-space or a T_2-space then Y is a semi-T_2 space.

The proof of the following theorem follows directly from the definitions of semi-connected spaces and totally semi-open maps.

Theorem 13. If $f : X \to Y$ is a totally semi-open bijection and Y is a semi-connected space, then X is connected.

Theorem 14. If $f : X \to Y$ is a totally semi-open bijection and Y is a semi-compact space, then X is compact.

Proof. Suppose that X is not compact, then there exists an open cover $\{U_i\}_{i\in I}$ of X which has no subcover. Since f is a totally semi-open bijection, then $\{f(U_i)\}_{i\in I}$ is a semi-open cover of Y and has not subcover, so Y is not semi-compact.

Conclusion

In the study, we have introduced the concept of totally semi-open functions in topological spaces and investigated several properties. Other properties of totally semi-open functions can be studied and totally semi-closed maps might also be defined and investigated.

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