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Parametric effect on discrete dynamic systems causing of chaos and dispersion

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التأثير البارامتري على النظم الديناميكية المتقطعة واحدات فوضى وتشتت

الملخص

النظام الديناميكي هو النظام الذي يتطور مع مرور الوقت، وهذا الوقت يمكن ان يكون متصل او متقطع في هذا البحث نحن نركز كليا على الوقت المتقطع حيت تم التوصل الى أفكار مثيرة للاهتمام بسهولة اكتر في الماضي المعادلة اللوجستيكية كانت تتضمن اعداد حقيقية ونلاحظ انها لاتنتج تشعبات وفوضى، في هذا البحث نحن نقدم نظام متماثل من معادلة لوجستيكية معقدة ودرسنا السلوك البارامتري لنظام ديناميكي متقطع لمتغيرمعقد. ودرسنا كذالك بعض الخواص الديناميكية كا نقط التوازن والاستقرار والتشعبات وكذالك الفوضى كما قدمنا النتائج العددية التي توكد التحليل العددي والنتائج ولقد استخدمنا برنامج المتلاب لتوضيح التشعبات والفوضي من خلال الرسومات.

Abstract:

In the past the logistic equation where applied to real numbers and produes no dispersion and chaos, in this paper, we present the equivalent system of complex logistic equation. We will study parametric behavior on discrete dynamic system of complex variable, and study some dynamic properties such as fixed points and their asymptotic stability, Lyapunov exponents, chaos and bifurcation. Numerical results which confirm the theoretical analysis are presented. It is noted that dispersion and chaos have accured. We have used AL-Matlab to clarify the bifurcation diagrams and chaos both in 2D and 3D.

Keywords: logistic equation, fixed points, stability, Lyapunov exponent, bifurcation, chaos, chaotic attractor.

Introduction:

A dynamical system can be continuous or discrete, these systems can be applied to real and complex numbers.
Consider the logistic equation

$$x_{n+1} = \rho x_n(1 - x_n) \quad n = 0,1,2,\dots \quad (1)$$

Where ρ is complex number $\rho = a + ib$, then the Logistic equation (1) will be of complex variables

$$Z_{n+1} = \rho Z_n(1 - Z_n) \quad n = 0,1,2,\dots \qquad (2)$$

And we call it a complex discrete dynamical systems.
In [4] the authors studied the dynamic properties of the Logistic equation (2).
Our aim of this paper is to study the dynamic behavior of

$$Z_{n+1} = \rho\, Z_n\, (1 - Z_n\bar{Z}_n), \qquad n = 0,1,2,\dots \quad (3)$$

Where $Z_n = x_n + iy_n$; x,y ∈ R , with initial conditions $Z_0 = x_0 + iy_0$, $|Z| \leq 1$.
We will study some dynamic properties such as fixed points, stability, bifurcation, and chaos. Through these properties we will explain the parametric temperature on the equation (3) in two cases ρ is real and ρ is complex.

The system with real parameter:
First, we must convert the equation (3) to system of equations as follow:

$$x_{n+1} + iy_{n+1} = \rho(x_n + iy_n)(1 - (x_n + iy_n)(x_n - iy_n)).$$

which gives

$$\begin{cases} x_{n+1} = \rho\ x_n(1 - x_n^2 - y_n^2), \\ y_{n+1} = \rho\ y_n(1 - x_n^2 - y_n^2). \end{cases} \qquad (4)$$

1-Fixed points and stability:
The fixed points of the equation (3) is the solution of the system (4) ,[see 6]

$$\begin{cases} x = \rho\, \mathrm{x}\,(1 - x^2 - y^2), \\ \mathrm{y} = \rho\ \mathrm{y}(1 - x^2 - y^2). \end{cases}$$

Then:
Let $x = 0$ we get

$$\begin{cases} 0 = \rho(0)(1 - 0 - y^2), \\ y = \rho y(1 - 0 - y^2). \end{cases}$$
$$1 = \rho(1 - y^2)$$

Then we get

$$y_{1,2} = \pm\sqrt{1 - \frac{1}{\rho}}$$

Now let $y = 0$ for the system we get

$$\begin{cases} x = \rho x(1 - x^2 - 0) \\ 0 = \rho(0)(1 - x^2 - 0) \end{cases}$$

Then we get

$$x_{1,2} = \pm\sqrt{1 - \frac{1}{\rho}}$$

Thus:
we get four fixed points we choose some fixed points, namely

- $fix_1 = (0,0)$,
- $fix_2 = \left(\sqrt{1-\frac{1}{\rho}}, 0\right)$,
- $fix_3 = \left(-\sqrt{1-\frac{1}{\rho}}, 0\right)$.

By considering a Jacobian matrix for each of these fixed points and calculating their eigenvalues , we can investigate the stability of each fixed point based on the roots of the system characteristic equation. (see [3])
The Jacobian matrix is given by

$$J=\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \rho - 3\rho x^2 - \rho y^2 & -2\rho xy \\ -2\rho xy & \rho - \rho x^2 - 3\rho y^2 \end{pmatrix}$$

The Jacobian matrix at $fix_1 = (0,0)$ is

$$J_{(0,0)} = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}.$$

The eigenvalues of this matrix are given by

$$|J - \lambda I| = 0 = \begin{vmatrix} \rho - \lambda & 0 \\ 0 & \rho - \lambda \end{vmatrix}.$$

Thus the characteristic equation reads

$$P(\lambda) = \lambda^2 - 2\rho\lambda + \rho^2 = 0$$

and has the roots

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2\rho \pm \sqrt{4\rho^2 - 4\rho^2}}{2}$$

$$= \rho.$$

Thus the first fixed point is stable if

$|\lambda_i| < 1 \quad (i = 1,2)$ [see 5], this mean that:
$|\rho| < 1$ that is $-1 < \rho < 1$.
While the $fix_2 = \left(\sqrt{1-\frac{1}{\rho}}, 0\right)$ yields the following characteristic equation

$$F(\lambda) = \lambda^2 + (2\rho - 4)\lambda - (3 - 2\rho) = 0.$$

Then the second fixed point is stable if $|\lambda_1| < 1$;
$|\lambda_2| < 1$, That is $2 > \rho > 1$
It is pretty clear that the $fix_3 = \left(-\sqrt{\frac{\rho-1}{\rho}}, 0\right)$ yields the same characteristic equation. So, the fix_3 is stable if $2 > \rho > 1$.

2- Lyapunov exponent

Since the Lyapunov exponent is a good indicator for existence of chaos, the Lyapunov characteristic Exponents (LCEs) play a key role in the study of nonlinear dynamical systems and they are measure of the sensitivity of the solutions of a given dynamical system to small changes in the initial conditions. One feature of chaos is the sensitive dependence on initial conditions; for a chaotic dynamical system at least one LCE must be positive. Since for non-chaotic systems all LCEs are non-positive, the presence of a positive LCE has often been used to help determine if a system is chaotic or not.[see 7]

Figure (1) shows the LCEs for the system (4) with the parameter values ρ and initial condition (x_0, y_0) = (0.4, 0) , We find that LCE1 = 1.0969 and LCE2 = 0.0082 .

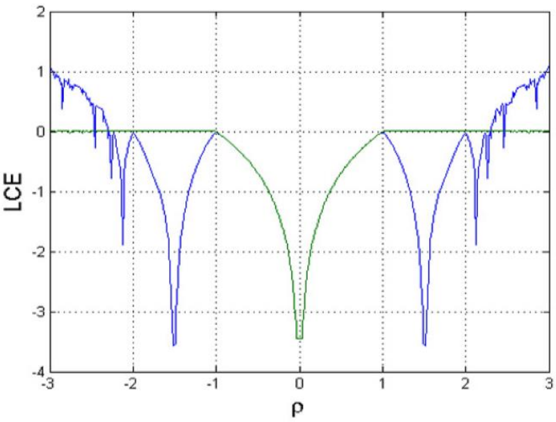


Figure 1: Lyapunov exponent of the system (4).

3- Bifurcation and chaos

In this section,the numerical experiments show the dynamical behavior of the discrete dynamical system (4) as follows:

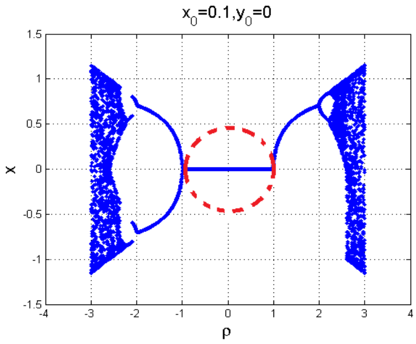


Figure2: Bifurcation diagram 0f (4) x vs. ρ.

We see clearly in Figure (2) the fixed point is stable at ρ between [-1,1], and stable also from ρ between [1,2], the bifurcation from a stable fixed point to a stable orbit of period (2) at $\rho = 2.2$,and then the bifurcation from period two to period four at ρ between [2.4,2.5]. The further period doubling occur at decreasing increments in ρ and the orbit becomes chaotic for $\rho \cong 2.6$.

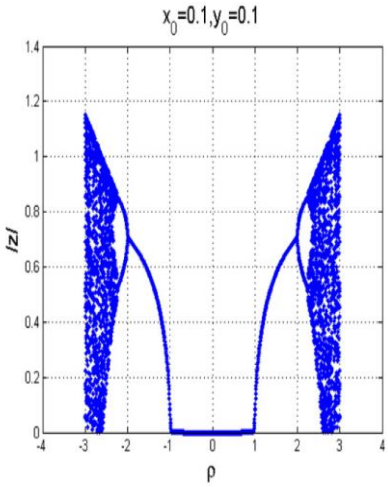


Figure4: Bifurcation diagram 0f (4) $|Z|$vs. ρ.

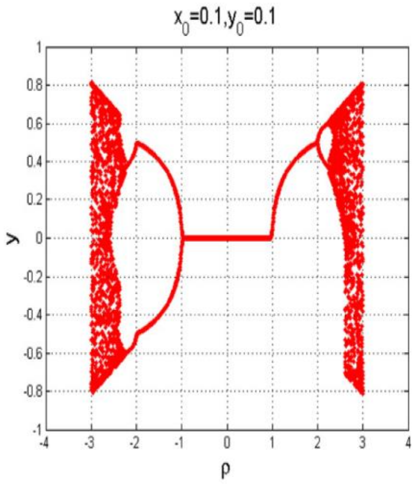


Figure3: Bifurcation diagram 0f (4) y vs. ρ.

Figure (3) shows the bifurcation diagram of y versus ρ, while Figure (4) shows the bifurcation diagram of $|Z|$ versus ρ.

4- Chaotic attractor:

In this section we are interested in studying the chaotic attractor for the system (4) .

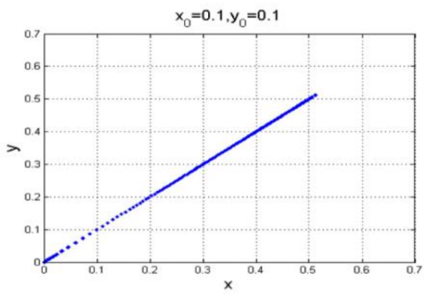


Figure 5: chaotic attractor of (4).

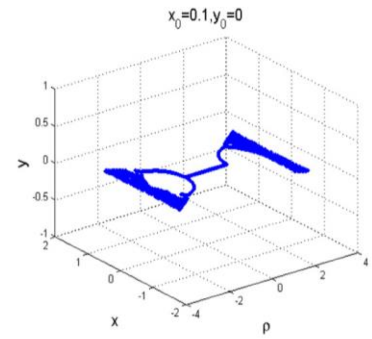


Figure 6: Bifurcation diagram 0f (4) in 3D

The System with complex parameter:

The system (3) with complex parameter $\boldsymbol{\rho = a + ib}$, where a , b ϵR can be written on the form:

$$\begin{cases} x_{n+1} = ax_n(1 - x_n^2 - y_n^2) + by_n(-1 + x_n^2 + y_n^2), \\ y_{n+1} = ay_n(1 - x_n^2 - y_n^2) + bx_n(1 - x_n^2 - y_n^2). \end{cases} \quad (5)$$

1-Fixed points and their stability:

When solving the system (5) we get a circle with center (0,0) and radius $\sqrt{1 - \frac{a}{a^2+b^2}}$).

Then the system has infinite number of fixed points located inside the circle and satisfies stability condition.

2- Lyapunov exponents:

Figure (7) shows the LCEs for the system (5) with the parameters $\rho = a + ib$ and initial conditions (x_0, y_0) = (0.2, 0), we find that LCE1 = 0.6778 and LCE2 = 0.0031 .

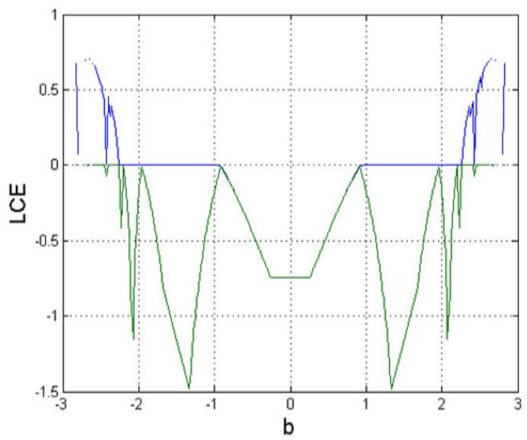
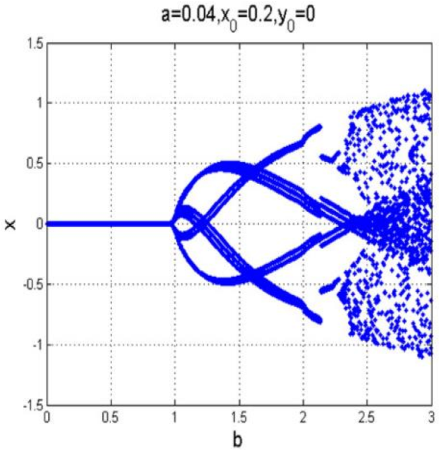


Figure 7: Lyapunov exponent of the system (5) where a=0.4.

3-Bifurcation and chaos:

In this section, The numerical experiments show the dynamical behaviors of the system (5):



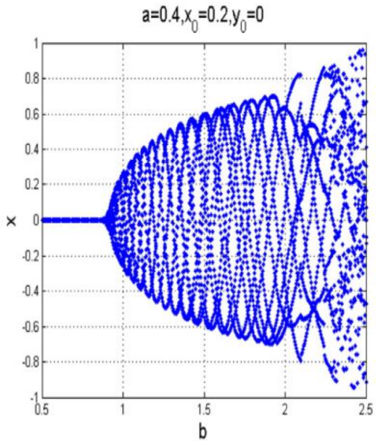


Figure8: Bifurcation diagram 0f (5) x vs. b Figure 9: Bifurcation diagram 0f (5) x vs. b.

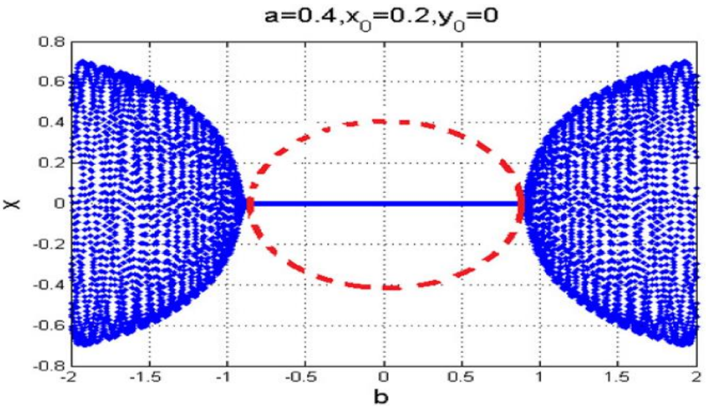
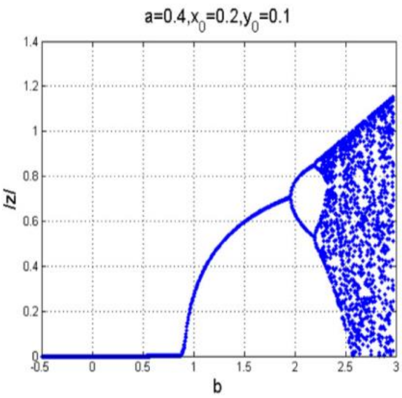


Figure10: Bifurcation diagram 0f (5) x vs. b.

we see clearly in Figure (10) the stability of fixed points are inside the circle , and the bifurcation from a stable fixed point to a stable orbit of period 2 at b=1.1 , and we note that from b=1.2 to 2.3 the bifurcation type is called hopf bifurcation , and the orbit becomes chaotic for b≅ 2.4.



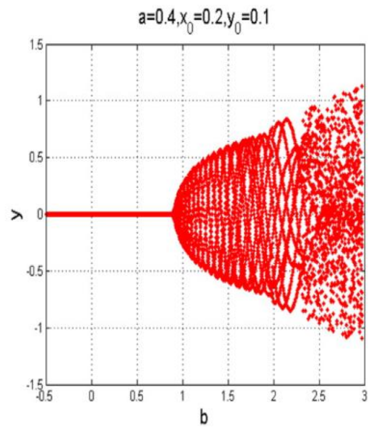


Figure13: Bifurcation diagram 0f (5) $|Z|$vs. b. Figure12: Bifurcation diagram 0f (5) y vs. b.

Figure (12) shows the bifurcation diagram of y versus b, while Figure (13) shows bifurcation diagram of $|z|$ versus b .

4- Chaotic attractor

In this section we are interested in studying the chaotic attractor for the system (5) .

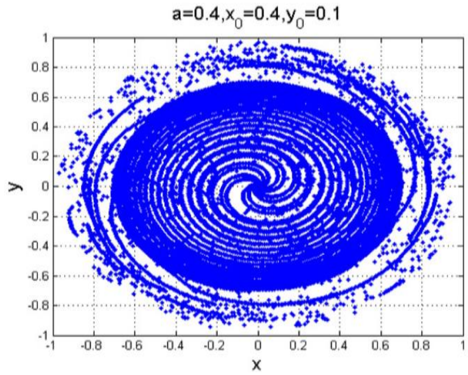


Figure14: chaotic attractor of (5).

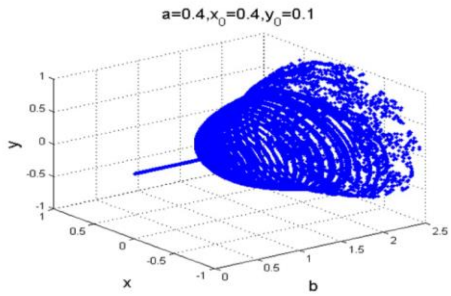


Figure15: Bifurcation diagram 0f (5) in 3D.

Conclusion:

The stability of these fixed points depends on the values of the parameter. Bifurcation diagrams as well depend on the values of the parameter. The discrete dynamical system with real parameter has different dynamic behavior from that system with complex parameter, the stability region of the system shrinked in case of complex parameters**.**

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