Semi-perfect Sets in Topological Spaces

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Abstract
In this paper, we introduce a new concept called semi-perfect sets in topological spaces. We define semi-isolated points and semi-perfect sets. Further, some interesting results and properties of these concepts are investigated.

Keywords: Semi-perfect set, Semi-isolated point, Semi-closed set, Semi-open set.

الملخص
في هذا البحث نقدم مفهوماً جديداً و هو مفهوم المجموعات شبه التامة في الفضاءات التبولوجية. قدمنا في البداية التعريف و الخواص للنقاط شبه المعزولة، ثم درسنا المجموعات شبه التامة مع برهنة العديد من الخواص و النظريات لهذه المجموعات.

الكلمات المفتاحية : مجموعة شبه تامة، نقاط شبه معزولة، مجموعة شبه مفتوحة، مجموعة شبه مغلقة.

1. Introduction

The concept of semi-open sets and semi-continuity was first introduced and investigated by Levine [4] in 1963. Crossley and Hildebbrand in [1] and [2] developed many related concepts in this area such as semi-closed sets, semi-closure, semi-interior and semi-homeomorphism. S. N. Maheshwari and R. Prasad in [5] introduced the concept of semi-operation axioms. T. M. Nour in [6] introduced the concept of totally semi-continuous functions as a generalization of totally continuous functions.

In this paper, we introduce and investigate the concept of semi-isolated points and semi-perfect sets in topological space; we prove some interesting results in this connection.

2. Preliminaries

This section contains some preliminary definitions and results which will be used throughout our paper.

Definition 1. Let A be a subset of a topological space X. A is said to be:

1) semi-open set [4] if $A \subseteq \overline{A^{\circ}}$, i.e., if there exists an open set U in X such that $U \subseteq A \subseteq \overline{U}$.

2) semi-closed set [2] if A is the complement of a semi-open set.

Definition 2. [2] The semi-closure of a set A in X is the intersection of all semi-closed sets that contains A; this set is denoted by $\underline{A}$.

Definition 3. [2] The semi-interior of a set A in X is the union of all semi-open sets of X contained in A; this set is denoted by $A_{\circ}$

Definition 4. [2]. A function $f: X \rightarrow Y$ is said to be:

1) irresolute if the inverse image of every semi- open set in Y is semi-open in X.
2) pre-semi-open if the image of every semi- open set in X is semi-open in Y.

Definition 5. [2] A bijection $f: X \rightarrow Y$ is said to be semi-homeomorphism if f is irresolute and pre-semi-open.

Definition 6. [2] A property which is preserved under semi-homeomorphism is said to be semi-topological property.

It is well-known that a point $x \in X$ is said to be an isolated point of a subset S of X if $x \in S$ and $\{x\}$ is open in S [3].

Definition 7. [3] Let A be a subset of X. A is said to be perfect if it is closed and has no isolated points.

3. Semi-isolated Points of a Set

In this section, we introduce the concept of semi-isolated points of a subset in a topological space. We have the following definition.

Definition8. A point $x \in S$ is said to be a semi-isolated point of S if there exists a semi-open set in X which contains x and does not contain any other points of S; so $\{x\} = S \cap U$ for some semi-open set U in X.

Obviously, every isolated point of a set S is also semi-isolated point of S, but the converse is not true in general, this showed by the following examples.

Example 1. In the space $\mathbb{R}$ with the natural topology, if $A = (0,1]$ then 1 is a semi- isolated point of A because $\{1\} = A \cap [1,2)$ and $[1,2)$ is semi-open. Note that 1 is not an isolated point of A.

Example 2. Let $X = \{a, b, c, d\}$ and let $\tau = \{\phi, X, \{a, b\}, \{b, c, d\}, \{b\}\}$.

Then d is a semi-isolated point of the set $\{c, d\}$ but d is not an isolated point of this set.

Remark1. If $A, B \subseteq X$ and if b is a semi-isolated point of both A and B, then:

1) b is a semi-isolated point of $A \cap B$.

2) it is not true in general that b is a semi-isolated point of $A \cup B$ as shown by:

Example 3. In the space $\mathbb{R}$, if $b \in \mathbb{R}$, then b is a semi-isolated point of both $A = (a,b]$ and $B = [b,c)$, $a < b < c$, but b is not a semi- isolated point of $A \cup B$.

Remark 2. If neither A nor B has semi-isolated points, then $A \cap B$ might have semi-isolated points as shown by

Example 4. Let $X = \{1,2,3,4,5\}$ and let $\tau = \{\phi, X, \{1,2,3\}, \{1,2,3,4\}\}$.

If $A = \{1,2,5\}$, $B = \{1,3,5\}$, then neither A nor B has semi-isolated points, but $A \cap B$ has a semi-isolated point 1.

Remark 3. If a is a semi-isolated point of S and $a \in Y \subset S$, then a is a semi-isolated point of Y.

Remark 4. In a topological space X, a point is semi-isolated of X if and only if it is an isolated point of X.

4. Semi-perfect Sets in Topological Spaces

In this section, we define semi-perfect sets in topological spaces. Further, we study some properties of semi-perfect sets and prove some results.

Definition 9. A subset S of a topological space X is said to be semi-perfect if it is semi-closed and has no semi-isolated points.

Example 5. The set $S = (0,1)$ is semi-perfect in $\mathbb{R}$, but the set $A = (0,1]$ is not semi-perfect, since 1 is a semi-isolated point of A.

Example 6. Let $X = \{1,2,3,4\}$ and $\tau = \{\phi, X, \{1,3\}, \{4,2\}\}$. Then $A = \{2,4\}$ is a semi-perfect set in X.

Remark 5. If S is semi-perfect in a space Y and $S \subseteq W \subseteq Y$ then it is false in general that S is a semi-perfect in the subspace W as shown by:

Example 7. In the space $\mathbb{R}$, let $A = (0,1)$, so $A \subset (0,1] \subset \mathbb{R}$. A is semi-perfect in $\mathbb{R}$, but A is not semi-perfect in the subspace $(0,1]$ since the complement of A in $(0,1]$ is not semi-open, so A is not semi-closed in $(0,1]$.

Remark 6. It is not true in general that the arbitrary union of semi-perfect sets is a semi-perfect set as shown by the following example.

Example 8. In the space $\mathbb{R}$, if $A = (-\infty, 0)$, $B = (0, \infty)$, then both A and B are semi-perfect in $\mathbb{R}$, but $A \cup B$ is not a semi-perfect set in $\mathbb{R}$ since it is not semi-closed.

Lemma 1. [4] Let $\{X_i\}_{i=1}^{n}$ be a collection of topological spaces and for each i, A_i is a semi-open set in X_i, then $A = \prod_{i=1}^{n} A_i$ is semi-open in the product space $\prod_{i=1}^{n} X_i$.

Lemma2. Let X be a topological space and $A \subset X$, if $\prod_{i=1}^{n} A$ is semi-open in the product space $\prod_{i=1}^{n} X$ then A is semi-open in X.

Proof. If $\prod_{i=1}^{n} A$ is semi-open in $\prod_{i=1}^{n} X$, then $\prod_{i=1}^{n} A \subseteq \overline{(\prod_{i=1}^{n} A)^{\circ}} = \prod_{i=1}^{n} \overline{A^{\circ}}$. It follows that $A \subseteq \overline{A^{\circ}}$ and A is semi-open.

Theorem 1. Let X be a topological space, $A \subset X$ and $S = \prod_{i=1}^{n} A$, if S is semi-perfect in the product space $\prod_{i=1}^{n} X$ then A is semi-perfect in X.

Proof. If S is semi-closed in $\prod_{i=1}^{n} X$, then Lemma 2 implies that A is semi-closed in X. Now, suppose that A has a semi-isolated point a. So, $\{a\} = A \cap U$, where U is semi-open in X and hence $\{(a, \dots, a)\} = \prod_{i=1}^{n} A \cap \prod_{i=1}^{n} U$. From Lemma 1, $\prod_{i=1}^{n} U$is semi-open and hence $\{(a, \dots, a)\}$ is a semi-isolated point of $\prod_{i=1}^{n} A$. Therefore, if S has no semi-isolated point in $\prod_{i=1}^{n} X$ then A has no semi-isolated point in X, so the proof is complete.

Suppose $f: X \to Y$ is a function and A is a semi-perfect set in Y, now we will study some conditions of the function f under which the set $f^{-1}(A)$ is also semi-perfect in X.

Remark 7. If $f: X \to Y$ continuous and if A is a semi-perfect set in Y, then $f^{-1}(A)$ not necessary to be semi-perfect in X as shown by:

Example 9. Let $f: (0,1] \to \mathbb{R}$ be the function defined as $f(x) = x$, where $(0,1]$ considered as subspace of $\mathbb{R}$; so f is continuous. If $A = (0,1)$, then A is semi-perfect in $\mathbb{R}$ but $f^{-1}(A) = A$ is not semi-perfect in $(0,1]$.

The following two results will be used later, their proofs can be found in [2].

Theorem 2. [2] If $f: X \to Y$ continuous and open then f is irresolute and pre-semi-open.

Theorem 3. [2]If $f: X \to Y$ is a homeomorphism then f is also a semi-homeomorphism.

Now, we prove our result.

Theorem 4. If $f: X \to Y$ is a continuous and open surjection and A is a semi-perfect set in Y then $f^{-1}(A)$ is a semi-perfect set in X.

Proof. Let $f: X \to Y$ be continuous and open, Theorem 2 implies that f is irresolute and pre-semi-open. Hence, $f^{-1}(Y - A) = X - f^{-1}(A)$ is semi-open in X and $f^{-1}(A)$ is semi-closed. Now, Suppose for a contradiction that $f^{-1}(A)$ has a semi-isolated point x, so $\{x\} = f^{-1}(A) \cap U$ for some semi-open set U in X.

Hence, $\{f(x)\} = A \cap f(U)$, where $f(U)$ is semi-open since f is pre-semi-open. It follows that $f(x)$ is a semi-isolated point of A, which is a contradiction.

Now, we will prove that semi-perfect is a semi-topological property and also topological property.

Theorem 5. If $f: X \to Y$ is a semi-homeomorphism and if A is a semi-perfect set in X then $f(A)$ is a semi-perfect set in Y.

Proof. From the definition of semi-homeomorphism, if A is semi-closed in X then $f(A)$ is semi-closed in Y. Now, Suppose that $f(A)$ has a semi-isolated point x, so $\{x\} = f(A) \cap U$ for some semi-open set U in Y. Hence, $\{f^{-1}(x)\} = A \cap f^{-1}(U)$ and $f^{-1}(x)$ is semi-isolated point of A.

The proof of the following theorem follows immediately from Theorem 3 and Theorem 5.

Theorem 6. If $f: X \to Y$ is a homeomorphism and if A is semi-perfect in X then $f(A)$ is semi-perfect in Y.

Conclusion

In the study, we have introduced the concept of semi-perfect sets in topological spaces and proved several results. More properties of semi-perfect sets can be investigated. Other concepts on semi-open and semi-closed sets might also be studied.

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