ISSUE 11 June 2021
مجلة العلوم الإنسانية والتطبيقية
Journal of Humanitarian and Applied Sciences
العدد 11 يونيو 2021

Differentiable and analyticFunction spaces

N. M.BEN. Youisf
University of Tripoli

Abstract:

In this paper we shall prove a theorem which is very important for the structure of complete analytic vector fields, and some lemmas. .

Keywords: differentiable, analytic function, analytical mapping and the class of C^{∞}differentiable functions.

الملخص

في هذه الورقة سوف نثبت نظرية مهمة جدا لبنية حقول المتجهات التحليلية الكاملة،وكذلك بعض البراهين.

1- Introduction :

We know from the definition of differentiable function of class C^{∞} that we can expand it in power series .

For (real) analytic in single variable are necessary but not sufficient condition that a function be (real) , analytic that can be expanded in a power series at each point a € U where U is an open set of R^N is that to be in C^{∞} (u) if f is real analytic on U we say that {f € C^{ω} (u) }.

A although knowledge of analytic function is helpful , since C^{ω} implies C^{∞} _ to know that any linear function f(x) = $\sum a_i x^i$, or polynomial p $(x^1,..., x^n)$

N variables is analytic function on u= R^n , the some is true for any quotient of polynomials (rational functions) if we exclude from the domain the points at which the denominator is zero.

Thus , for example a determinant is an analytic function of it's entries and , if we exclude nxn matrices of determinant zero ,(which have no inverse)then each entry in the invers A^{-1} of Matrix a is an analytic and (hence C^{∞}) function of the entries in matrix A.

Mapping:

Let R^m and R^n denote two Euclidean spaces of m and n dimension, respectively . Let O and O′ be open subsets, O $\subset$ R^m , O′ $\subset$ R^n and suppose φ is mapping of O in to O′.

The mapping , φ is differentiable if the coordinates y_j (∅ (P)) of ∅ $(p)$are diffentiable (that is , indefinitely differentiable)functions of the coordinates x_i (p), pϵo.

The mapping φ is called analytic if for each point pϵo there exists a nigh- bor hood U of P and n power series pj (1≤ j ≤ n) .

m variables such that y_j (φ (q)) =P_j (x_1 (q) – x_1(p),……, x_m(q) – X_m(P)),

(1≤ j ≤ n) for qϵ U .

A differentiable mapping ∅: $O \rightarrow O'$ is called a diffeomorphism of O on toO' if φ (O)=O', ∅ is one – to- one and inverse mapping $\emptyset^{-1}$ is differentiable.

When n = 1 it is customary to replace term mapping by term function.

An analytic function on R^m which vanishes on an open set is identically zero.

For differentiable function the situation is completely different .

In fact , if A and B are disjoint Sub sets of R^m , A compact and B closed then there exists differentiable function ∅ which is identically 1 on A and identically 0 on B.

Example: let 0 < a < b and consider the function f on R defined by

$$f(x) = \begin{cases} \exp\left(\frac{1}{x-b} - \frac{1}{x-a}\right) & \text{if } a < x < b \\ o & \text{otherwise} \end{cases}$$

Then f is differentiable and the same holds for the function $F(x)= \frac{\int_x^b f(t)dt}{\int_a^b f(t)dt}$

Which has value 1 for x ≤ a and 0 for x ≥ b .

Example: the functionΨ on R^m given by

Ψ (x_1,….., x_m) = f (x^2_1 + …..+ x^2_m) is differentiable and has values 1 for

x^2_1 +…..+ x_m^2 ≤ aand O zero for x_1^2+…….+ x_m^2 ≥ b .

let S and S' be two concentric spheres in R^m.

S' lying inside S. starting from Ψ we can by means of transfrom of R^m constarcut a differential function on R^m.

With value in interiors' and value 0 outside S. ⊂

Turning now to the sets A and B we can owing to the compactness of A, find finitely many spheres Si (1 ≤ i ≤ n) such that the corresponding open balls Bi (1≤

$i \leq n$) form covering A (that is, $A \subset \cup_{i=1}^{n} Bi$) and such that the closed ball $\overline{Bi}$ ($1 \leq i \leq n$) do not intersect B.

Each sphere Si can be shrunk to concentric sphere S_i' , such that the corresponding open balls B_i' still covering of A.

Now let Ψ_i be differentiable function on R^m which is identically1 on B_i' an identically 0 in the complement of Bi.

Then the function :

$\Psi = 1-(1-\Psi_1)(1-\Psi_2)\ldots\ldots(1-\Psi_n)$

Is differentiable function on R^m which identically 1 on A and identically 0 onB .

Function of class C^∞ and real analytic function let us say that f of class C^∞ if f is of class C^q for every q. If f is of class C^∞ and $\lim_{q\to\infty} R_q(x) = 0$, them in place of Taylor's.

Formula with remainder we may put the corresponding infinite series .

This infinite series is called the Taylor series for f(x) at x_0.

If K is convex subset of D and $x_0 \in K$ then the following is a sufficient condition that f(x) be the sum of it's Taylor series for every $x \in k$.

Suppose that there is a positive number M whose qth.

Power bounds every qth-order partial derivative of f, namely, $\left|f_{i1,i2\ldots\ldots iq}^{(x)}\right| \leq M$ for every $x \in k$, q=1,2,..., and $1 \leq i_1,.., i_q \leq n$

Then $c=m^q$ where $\left|R_q(x)\right| \leq cn^{q/2}|h|^q$.

$h= x-x_0$.

$\left|R_q(x)\right| \leq \frac{M^q n^{q/2}|h|^q}{q!} = \frac{B^q}{q!}$ Where $B = mn^{1/2}|h|$.

Since $\frac{B^q}{q!} \to 0$ as $q \to \infty$ $lim_{q-\infty} R_q(x) = 0$ For every $x \in k$.

A function is called analytic if every $x_0 \in D$ has a neighborhood U_{xo} such that the Taylor series at to x_o converges to f(x) for every $x \in Uxo$.

We have proved the following: Let f be of Calls c^{oo}, and Suppose that every $x_o \in$ D has a neighborhood U_{xo} in which an estimate$\left|f_{i1,i2\ldots\ldots iq}^{(x)}\right| \leq M^q$holds Then f is analytic.

The positive number M may depend on X_o and on radius of U_{xo}.

2- OUR main result

2.1 proposition the function f:iR → R defend on R by

$$f(s) = \begin{cases} 0\,, s \le 0 \\ \exp(-1\backslash s)\,,\ s > 0 \end{cases}$$ is a C^∞ function .

Proof : Assume that , for some integer n the nth derivative of f is defined $f^{(n)}(s)$ $= \begin{cases} \exp(\frac{-1}{s})pn(\frac{1}{s}) \\ 0 \quad\quad if\ s < 0 \end{cases}$ where p_n is some polynomial if s > 0

By differentiation

$$f^{(n+1)}(s) = \begin{cases} \exp\left(\frac{-1}{s}\right) P_{n+1}(\frac{1}{s})\ if\ \ s > 0 \\ 0 \quad\quad if\ s < 0 \end{cases}$$

To find $f^{(n+1)}(0)$ we use the fact that for any integer N ≥ 0.

$$\lim_{s \to o+} \left\{ \frac{1}{s^N} \exp\left(\frac{-1}{s}\right) \right\} = 0$$

it follows that

$$\lim_{s \to o+} \left(\frac{f^n(s) - f^n(0)}{s}\right) = \lim_{s \to o+} \left\{ \frac{1}{s} \exp\left(\frac{-1}{s}\right) pn\left(\frac{1}{s}\right) \right\} = 0$$

$$\lim_{s \to o-} \left(\frac{f^n(s) - f^n(0)}{s}\right) = \lim_{s \to 0-} \left\{ \frac{0}{s} \right\} = o$$

This implies that $f^{(n+1)}(0)=0$.

Our original assumption is true when n=0 and so, by induction , it is true for any positive integer **n.f** is there for a C^∞ function. **.**

2.2 lemma :

Let D = E^1 and let $f(x) = \begin{cases} \exp\left(\frac{-1}{x^2}\right) if\ x > o \\ 0 \quad if\ \ x \le o \end{cases}$

Show that, f ∈ $C^\infty(E^1)$?

Proof: let us show that this function is of class C^∞ and $f^q(0)$= 0 for every q=1,2,………,0 for x ≠ 0 , the derivativesf^q (x) can be compute by elementary calculus, and eachf^q continuous on E^1 - {0}.

It is at the point o where f must be examined . Now

$* \lim_{k \to +\infty} (u^k \exp(-u)) = 0$ for each k=1,2,…. a Fact that we prove immediately below.

If x <0 , then f(x) = f '(x) = f "(x) =0…….=0.

With k =0 , exp ($\frac{-1}{x^2}$) →0 as x → o$^+$

Since f(0) = 0 , f is continues . If x > 0 , f '(x) = $\frac{2}{x^3}$ exp ($\frac{-1}{x^2}$) = 2x.$\frac{1}{x^4}$ exp ($\frac{-1}{x^2}$)

With k=2 , f '(x) → 0 as x→ 0^+

There for $\lim_{x \to 0}$ f ' (x) =0 , f '(0) =0 and f is of class c^1.

For each (q=2,3,…..,) , $f^q(x)$ is a polynomial in 1/x times exp (-1/x^2)

For x > 0 **.**

Hence $\lim_{x \to 0} f^q(x) = 0\,, by\ induction\ on\ q\,, f^q(0) = 0$

And f ∈ C^q for every q , thus f ∈ C^∞ . If we expand f by Taylor's formula about O, then f(x) = R_q(x) for every x.

If x > 0 the remaider R_q(x) does not tend to O as q → ∞ . Hence f is not an analytic function .

* $\lim_{k \to \infty} u^k$ exp (-u)= 0 for each k= 0,1,2,….. . for each u < 0 let Ψ (u) = u^{-k} exp u , then Ψ' (u)= (u - k) u^{-k-1} exp u.

$\Psi'' = [u^2 - 2ku + k(k+1)]u^{-k-2}$ exp u The express in brackets has minimum when u = k and is positive there .

Hence Ψ'' (u) > 0 for all u> o Let us apply Taylor's formula to Ψ , with q=2: Ψ(u) = Ψ(u_0) + Ψ' (u_0) (u-u_0) + $\frac{1}{2}\Psi''$ (v)(u-u_0)^2 with v between u and u_0 Since Ψ'' (u) > 0.

Ψ(u) ≥ Ψ(u_0) + Ψ' (u_0) (u-u_0) . If u_0> k , then Ψ' (u_0) > 0 and the righthand side tends to + ∞ as u → ∞.

Hence Ψ(u) → +∞ and $\frac{1}{\Psi(u)}$ → 0 as u → +∞. Which complete the proof **.**

2.3 Theorem. If P : IR^N → IR is a polynomial function and 0 ≠ Φ:

IR^k→ IR is an affine function such that P(q) = 0 for the points q of the hyper plane {q ∈IR^N : Φ(q) = 0} then Φ is a divisor of P in the sense that P = ΦQ with some (unique) polynomial Q : IR^N → IR.

Proof. Trivially, any two hyperplanes are affine images of each other.

In particular there is a one-to-one affine (i.e linear + constant) map- ping A : $IR^N \leftrightarrow IR^N$.such that {q ∈$IR^N$: Φ(q) = 0} = A({q∈ IR^N: X_1(q) = 0}). Then R:= Po A is a polynomial function such

that R(q)=0 for the points of the hyper plane {q ∈ IR^N: x_1(q) =0}.

We can write R=$\sum_{k1.........kN=0}^{d} \alpha_{K1.........,K_N} x_1^{k1} \dots x_N^{kN}$ with a suitable finite family of coefficients $\alpha_{K1.......KN}$By the Taylor formula,$\alpha_{K1.........K_N}$ =

$$\frac{\partial^{k1+\cdots..+kN}}{\partial x1^{k1......}\partial x_N^{kN}}\Big|_{X1=\cdots............=XN=0}^{R.} \quad it\ follows\ that\ \alpha_{K1.........K_{N=0}} \quad for\ k_1 > 0, since$$

R vanishes for x_1 = 0. This means that R= X_1 R_o with the polynomial Ro :=$\sum_{k1=1}^{d}\sum_{k2,......kn=0}^{d} x_1^{k1-1}x_2^{k2} \dots\dots\dots\dots x_N^{kN}$By the same argument .

That is Φ is the sum of a linear functional with a constant.

Applied for the polynomial function Φ of degree d=1 in place of R, we see thatΦ o A= αx_1 for some constant (polynomial of degree 0)α ≠ 0.

That is Φ = αx_1 o A^{-1}. Therefore

P = Ro A^{-1} = [x_1 R_0] o A^{-1} = (x_1 o A^{-1})(R_0 o A-1) = Φ($\frac{1}{\alpha}$$R_0oA^{-1}$). a Since the inverse of an affine mapping is affine as well, the function

Q:= ($\frac{1}{\alpha}$$R_0$o A^{-1}). is a polynomial which suits the statement of the theorem.

2.4 Theorem: Assume that G ⊂ $R^N$is an open connected set such that G∩E_o≠ ∅ , And let Φ: G ⟶ R be an analytic function such that Φ (x) = 0 for all $x_\in$ G∩E_0., Then Φ (x) = x_1Ψ(x) for some analytic function Ψ: G ↔R where x_1 = <x.e> and x = (x_1, ,x_2,…., x_N) ∈R_N.

<u>proof:</u> let E_0={ p∈ R^N : x_1(p) =0} , be a hyper-plane Φ(p)=0 for p∈ E_0.

$$\phi(p) = \sum_{k=1}^{\infty}, \sum_{n_1+\cdots+n_N=k} a_{n_1\dots n_N} x_1^{n1}(p) \dots x_N^{n_N}(p)$$

p∈ E_o ⟹ X_1(p)= 0, x_1^{n1} (p)…$x_N^{n_N}(p) = 0$, if $n_1 > 0$

0=$\Phi(p) = \sum_{k=1}^{\infty}\sum_{n2+\cdots+n_N=k} a_{n_0n_2\dots n_N} X_2^{n2}(p)\dots X_N^{n_N}(p)$,

By assumption .

$P = \xi_2 e_2 + \cdots + \xi_N e_N \in E_0 , \xi_2 , \ldots , \xi_N \in R , arbitrary.$

$0 = \Phi(p) = \sum_{n_2+\cdots+n_N=k} a_{0\ n_2 \ldots n_N} \xi_2^{n_2} \ldots \xi_N^{n_N}$

$$a_{0\ n_2 \ldots n_N} = \frac{\partial^{n_2+\cdots+n_N} \Phi(\xi_2 e_2 + \cdots + \xi_N e_2)}{\partial_{x_2}^{n_2} \ldots \partial_{x_N}^{n_N}} \frac{1}{n_2! n_N!} = 0. \quad a_{n_2 \ldots n_N} = 0, \forall n_2, \ldots , n_N .$$

$$\Phi(p) = \sum_{k=1}^{\infty} \sum_{\substack{n_1+\cdots+n_N=k \\ n_1>0}} a_{n_1 n_2 \ldots n_N} \mathrm{x}_1^{n_1}(p)$$

$$= x_1(p) \sum_{k-1}^{\infty} \sum_{\substack{n_1+\cdots+n_N=k \\ n_1>0}} a_{n_1 n_2 \ldots n_N} X_1^{n_1^{-1}}(p) \ldots X_N^{n_N}(p)$$

$$= X_1(p) \Psi(p).$$

$$\Psi(p) = \sum_{I=0}^{\infty} \sum\nolimits_{m=0}^{\infty} \sum_{n_2+\cdots+n_N=I} a_{n_1 n_1 \ldots n_N} X_1^m(p) \ldots X_N^{n_N}(p)\ with$$

$$m = n_1 - 1\ and\ L = k - 1$$

<u>Remark:-</u> we know that the function

$$\varphi(t) = \begin{cases} e^{-1/t} & if\ t > 0 \\ 0 & if\ t \leq 0 \end{cases}$$

Is infinitely differentiable , since also the function

$$R^m \longrightarrow R. \qquad x \longrightarrow 1 - |x|^2$$

Is smooth(i.e.. infinitely differentiable), it follows that the same is true for the composition of both functions , more precisely we have : c >0 and

$$\omega(x) = \begin{cases} ce^{1/(|x|^2-1)} & if\ |x| < 1 \\ 0 & if\ |x| \geq 1 \end{cases}$$

Then $\omega \in C^\infty(R^m, R), \omega \geq 0\ and$

$supp(\omega) = \overline{\{y \in R^m | \omega(y) \neq 0\}} = \bar{B}^m$

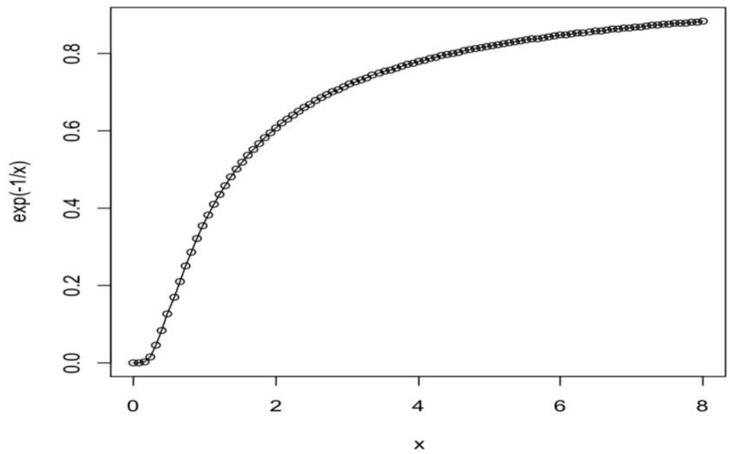
Now we choose

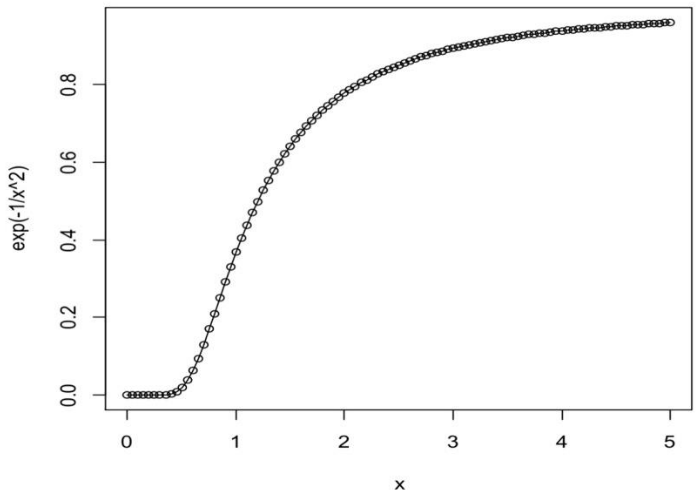
C= $\int_{-1}^{1} e^{1/(|x|^2-1)}\ dx$ *and for every* $\epsilon > 0$ *we set* $\omega_\epsilon(x) = \epsilon^{-m}\omega\left(\frac{x}{\epsilon}\right), \forall\, x \in R^m$.

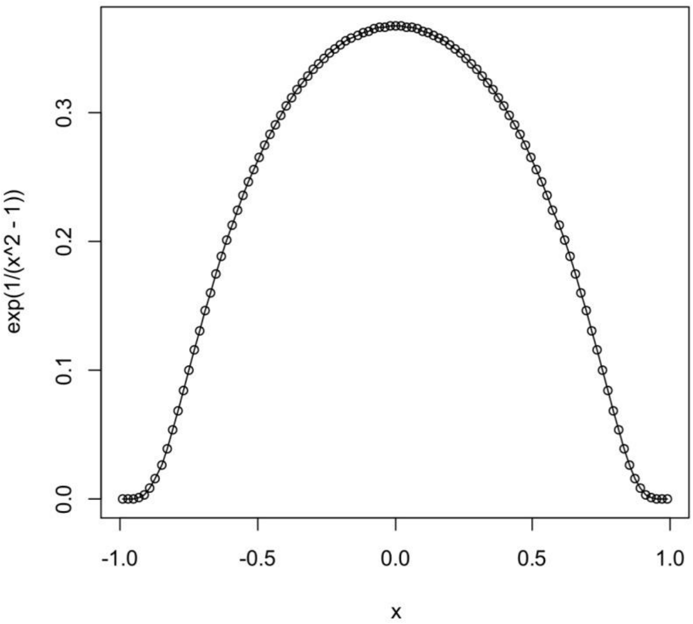
Then evidently we have

$$\omega_\epsilon \in C^\infty(R^m, R)\,, \omega_\epsilon \geq 0.\ supp\,(\omega_\epsilon) = \epsilon\bar{B}^m,$$
$$\omega_\epsilon(-x) = \omega_\epsilon(x), \quad \forall\, x\ \in R^m$$

$$\int_{R^m} \omega_\epsilon(x)dx = 1.$$







Reference

1. *N. M. BEN Yousif, Complete polynomial vector fields on Simplexes Electronic Journal of qualitative Theory of diffential Equation No.5 2004:. pp. (1-10) .*
2. *ROBERT.J.walker.Algebraic curues, Princeton-New Jersey 1950.*
3. *Confor mal mapping, Zeer Niehari United States of America, Dover Publication. New York, NY 10014.*
4. *4. K isaku Yosi da, Function analysis springer-verlag, Berlin Heidelberg New York 1980. 57*

5.Tucory and Special Functions , Willand Miller, Jr. New Yorhand London, 1968 .

6. Kosaku Yosida , Function analysis / 1980 .

7 . Real and Complex Analysis ./ 1987 .

8 . an introduction to differentiable Manifolds and Riemannian Geometry / 1975 .