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On some finitely generated subsemigroups of S(X).

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الملخص:

S(X) نصف زمرة مع عملية تركيب الدوال المتصلة من والى الفضاء التبولوجي X. سوف نبرهن لعدد من الفضاءات ان S(X) تحوي نصف زمرة جزئية كثيفة مولدة بعنصرين عندما تكون S(X) معرف عليها تبولوجي متراص-مفتوح.

Abstract: S(X) is the semigroup, under composition, of all continuous selfmaps of the topological space X. In this paper, we show for a number of spaces that S(X) contains a dense subsemigroup generated by two elements when S(X) given the compact-open topology.

Keywords: semigroup, clopen set, continuous selfmap, IN-absorbing space, compact-open topology.

1.Introduction

Let X be an infinite set. In [1] Sierpiński proved the following result:

Theorem 1.1. Every countable family $f_1, f_2, \dots : X \to X$ of maps can be generated by two such maps.

In terms of semigroups, Sierpiński proved that any countable subset of the semigroup τ_x, under composition, of all selfmaps on X is contained in a 2- generated subsemigroup of τ_x. A simpler proof was given by Banach in [8]. However, the result of Evans [10], published 17 years later, that any countable semigroup can be embedded in a 2- generated semigroup follows at once from Sierpiński's result. Higman, Neumann and Neumann in [4, Theorem IV] proved that every countable group is embeddable in a 2-generator group. 42 years later, Galvin in [3] proved that every countable set of permutations of X is contained in a 2- generated subgroup of the symmetric group S_X. However, this permutational analogue of Sierpiński's theorem implies theorem IV in [4]. In [2] Mitchell and Péresse proved that any countable set of surjective maps on an infinite set of cardinality $\aleph_n$ with n ∈ IN can be generated by at most $n^2/2 + 9n/2 + 7$ surjective maps of the same set; and there exist $n^2/2 + 9n/2 + 7$ surjective maps that cannot be generated by any smaller number of surjections. Moreover, in the same paper was presented that several analogous results for other classical transformation semigroups such as the injective maps, Baer – Levi semigroups and the Schützenberger monoids. Let X be equipped with a topological structure.

The symbol S(X) denotes the semigroup, under composition, of all continuous self- maps of the topological space X; let S(X) have the compact-open topology. It was shown in [9] there is a subsemigroup of S(X) generated by two maps, which is dense in S(X) when X is the rationals, the irrationals, the countable discrete space, the Cantor space or m-dimensional closed unite cube.

The main aim of this paper is, using an elementary argument which is different from the one in [9], to show for a number of spaces X such as L, Q × C and P × Q that S(X) contains a dense subsemigroup generated by two elements.

2. Definitions and theorems

2.1 Definition: A topological space X is said to be an IN-absorbing space if it is a disjoint union of countable many clopen homeomorphic copies of itself.

2.2 Theorem [11]. Let X be an IN-absorbing space. Then every countable subset of S(X) is contained in a 2- generated subsemigroup of S(X).

2.3 Corollary [9],[11]. Every countable subset of S(P), where P is the irrationals, is contained in 2- generated subsemigroup of S(P).

The following is equivalent to the definition 2.1

2.4 Lemma [11].A topological space X is an IN-absorbing space if and only if $X \simeq X \times$ IN.

Proof.(⇒)obvious.

(⇐) If $f : X \simeq X \times$ IN, then $f^{-1}(X\times\{i\})$ is non-empty clopen subset of X and it is as a subspace homeomorphic to $X = \bigcup\{ f^{-1}(X\times\{i\}) : i \in \text{IN}\}$.

2.5 Corollary [9] ,[11]. Every countable subset of S(Q) is contained in a 2-generated subsemigroup of S(Q).

2.6 Corollary[11]. Let X be an infinite discrete space. Then every countable subset of S(X) is contained in a 2-generated subsemigroup of S(X).

Proof. Since any two discrete spaces are homeomorphic iff they have the same cardinality, so $X \simeq X \times$IN and by lemma 2.4 and theorem 2.2 the proof is completed.

2.7 Corollary[11]. Let $L = C \setminus\{p\}$ for $p \in C$. Then every countable subset of S(L) is contained in a 2-generated subsemigroup of S(L).

2.8 Lemma. If X is any toplogical space, then IN × X is an IN-absorbing space.

Proof. It is clear that IN × X $\simeq$ (IN × IN) × X $\simeq$ IN ×(IN × X) and by lemma 2.4, we are done.

2.9 Lemma. If X is an IN-absorbing space and Y is any topological space, then X × Y is an IN-absorbing space.

Proof . Similar to proof of lemma 2.8

X × Y × IN ≃ X × Y × (IN × IN) ≃ (X × IN) × (Y × IN) ≃ X ×(Y × IN) ≃ (X × IN) × Y ≃ X × Y.

Clearly, the previous lemma can be generalized to the following:

2.10 Lemma. Let Y_i be a topological space for each $i = 1,2,3,\dots,n$ and Y_j is an IN-absorbing for some j. Then $X = \prod_{i=1}^{n} Y_i$ is an IN-absorbing space.

The spaces Q × C and P × Q are unique up to homeomorphism [5].

2.11 Corollary. Let X be one of the spaces Q × C , P × Q , Q × IR or P × IR . Then every countable subset of S(X) is contained in a 2- generated subsemigroup of S(X).

Proof. By lemma 2.9, we see that X is an IN-absorbing space and by theorem 2.2 we are done.

2.12 Theorem. Let X be the space in lemma 2.10 such that Y_i is second countable for all i and let S(X) have the compact – open topology. Then there is a subsemi-group of S(X), generated by two maps, which is dense in S(X).

Proof. We know that S(X) is separable because X is second countable [7]. If $\{f_n\}_{n=1}^{\infty}$ is the family that dense, then by corollary 2.10 and theorem 2.2 there exist two maps which generate a subsemigroup of S(X) containing $\{f_n\}_{n=1}^{\infty}$. Clearly, this subsemigroup is dense in S(X).

2.13 Corollary. Let X be the rationals, the irrationals, the countable discrete space, L, Q × C, P × Q, Q × IR or P × IR. Then there is a subsemigroup of S(X), generated by two maps, which is dense in S(X).

Proof. Obvious from the previous theorem.

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