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Some properties of Synchronization and Fractional Equations

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Abstract: In this paper we propose the definition of Mittag-Leffler stability and introduce the stabilization of nonlinear fractional order dynamic systems. Then we provide the fractional comparison principle. We partly extended the application of Riemann-Liouville fractional systems by using Caputo fractional systems. An illustrative example is provided as a proof of concept .We also studied certain properties of fractional calculus associated with Laplace transforms

Keywords: *Nonlinear dynamics, fractional calculus, fractional derivatives, stability analysis*

1. Introduction

Fractional calculus is generally believed to have stemmed from a question raised in the year 1695 by L'Hopital and Leibniz [1]. It is the generalization of integer-order calculus to arbitrary order one. Frequently, it is called fractional-order calculus, including fractional-order derivatives and fractional-order integrals. Reviewing its history of three centuries, we could and that fractional calculus was mainly interesting to mathematicians for a long time, due to its lack of application background. However, in the previous decades more and more researchers have paid their attentions to fractional calculus, since they found that the fractional-order derivatives and fractional-order integrals were more suitable for the description of the phenomena in the real world, such as viscoelastic systems, dielectric polarization, electromagnetic waves, heat conduction, robotics, biological systems, finance and so on Owing to great efforts of researchers, there have been rapid developments on the theory of fractional calculus and its applications, including well-posedness, stability, bifurcation and chaos in fractional differential equations and their control[2,3,4]. Several useful tools for solving fractional-order equations have been discovered, of which Laplace transform is frequently applied. Furthermore, it is showed to be most efficient and helpful in analysis and applications of fractional-order systems, from which some results could be derived immediately,the authors investigated stability of fractional-order nonlinear dynamical systems[5,6,7]. In this paper, we will discuss the Laplace transform of the Caputo fractional difference and the fractional discrete Mittag-Leffler functions and use the Laplace transform method to solve another kind of discrete fractional[8,9,10].

2. Preliminaries

We first recall some well known definitions about Mittag-Leffler function similar to the exponential function. These functions are frequently used in solving of the integer-order equations and fractional order equations:

2.1.Definition (Mittag-Leffler Function)[11]

Mittag-Leffler function is given by:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)} \qquad (1)$$

Where $\alpha > 0$. The Mittag-Leffler function with two parameters appears most frequently to have the following form:

$$E_{\alpha,\ \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)} \qquad (2)$$

where $\alpha > 0$ and $\beta > 0$. For $\beta = 1$, we have $E_{\alpha}(z) = E_{\alpha,\ 1}(z)$. Also, $E_{1,\ 1}(z) = e^{x}$. Moreover, the Laplace transform of Mittag-Leffler function in two parameters is represented as follows:

$L\{E_{\alpha,\ \beta}(-\lambda t^{\alpha})\} = \frac{s^{\alpha-\beta}}{s^{\alpha}+\lambda}\ ,\quad R(S) > |\lambda|^{\frac{1}{\alpha}}$(3)

Where s is the variable in Laplace domain, R(s) denotes the real part of the variable s , $\lambda \in$R and L{·} stands for the Laplace transform.

Corollary (2.1) : *Let* β ,γ ,δ , *ω∈C* , *R(γ)* >0, *R(β) > 0*, *R(δ)* >0. *Then*

$$\int_0^z t^{\gamma-1} E_{\beta,\ \gamma}^{\delta}(\omega t^{\beta})dt = z^{\gamma} E_{\beta,\ \gamma+1}^{\delta}(\omega z^{\beta}) \qquad (4)$$

In particular,

$$\int_0^z t^{\gamma-1} E_{\beta,\gamma}(\omega t^{\beta})dt = z^{\gamma} E_{\beta,\gamma+1}(\omega z^{\beta})(5)$$

and

$$\int_0^z t^{\delta-1} Q(\gamma,\delta;\omega t)dt = \frac{1}{\delta} z^{\delta} Q(\gamma,\delta+1;\omega t) \qquad (6)$$

Lemma 2.1[12]. Suppose b ≥ 0, β >0 and a(t)is a nonnegative function locally integrable on 0 ≤ t <T (some T ≤ +∞),

and suppose u(t) is nonnegative and locally integrable on

0 ≤ t <T with

$$\mathrm{u(t)} \leq \mathrm{a(t)} + \mathrm{b}\int_0^t (\mathrm{t}-\mathrm{s})^{\alpha-1}\mathrm{u(s)}\,ds$$

on this interval. Then

$$\mathrm{u(t)} \leq \mathrm{a(t)} + \theta\int_0^t \mathrm{E}_{\beta}'(\theta(\mathrm{t}-\mathrm{s}))\mathrm{a(s)}\,ds \qquad 0 \leq \mathrm{t} < \mathrm{T}$$

where

$$\theta = \frac{(\mathrm{b}\Gamma(\beta))1}{\beta} E_{\beta}(z) = \sum_{k=0}^{\infty} \frac{z^{k\beta}}{\Gamma(\mathrm{k}\beta+1)} \qquad \mathrm{E}_{\beta}'(z) = \frac{d}{dz} E_{\beta}(z)$$

$$\mathrm{E}_{\beta}'(z) \cong \frac{z^{\beta-1}}{\Gamma(\beta)} \text{as } z \to 0_{+} \qquad \mathrm{E}_{\beta}'(z) = \frac{1}{\beta} e^{z}\ z \to +\infty.$$

$$and\ E_{\beta}(z) = \frac{1}{\beta} e^{z}\ z \to +\infty. If\ \mathrm{a(t)} \equiv \mathrm{a, constant, then}$$

$\mathrm{u(t)} \leq \mathrm{a}E_{\beta}(\theta t)$

2.2. Definition (Riemann-Liouville fractional integral)[13]
The Riemann-Liouville fractional integral of the order $\alpha > 0$ of a function y: (0;1) →R , is given by:

$$I_{0_+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds, \qquad (7)$$

Provided that the right hand side of this equation is point wise defined on $(0,\infty)$.
2.3.Definition. Suppose that : $u(t, x): R_+ \times [0,L] \to R^m$ for some $m > 0$, where u is of class $\ell_2[0,L]$ with respect to x . Then $\|.\|_2$ of $u(t,x)$ is defined by

$$\|u(t,x)\|_2 = \left\{ \int_0^L \|u(t,x)\|^2 dx \right\}^{\frac{1}{2}}$$

where $\|.\|$ is Euclidean norm
2.4. Definition Riemann-Liouville and Caputo fractional operators[14]
Riemann-Liouville and Caputo fractional operators as the main tools in fractional calculus. The uniform formula of a fractional integral with $\alpha\epsilon(0,1)$ is defined as:

$$_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \qquad , \qquad (8)$$

Where $f(t)$ is an arbitrary integrable function $_aD_t^{-\alpha}$, and it is the fractional integral of order α on $[a,t]$, and $\Gamma(\cdot)$ denotes the Gamma function. For an arbitrary real number p, the Riemann-Liouville and Caputo fractional derivatives are defined respectively as:

$$_aD_t^p f(t) = \frac{d^{[p]+1}}{dt^{[p]+1}} \left[{_aD}^{-([p]-p+1)} f(t) \right] \qquad (9)$$

and

$$_a^cD_t^p f(t) = {_aD_t}^{-([p]-p+1)} \left[\frac{d^{[p]+1}}{dt^{[p]+1}} f(t) \right] \qquad (10)$$

where $[P]$ stands for the integer part of p, D, and ^cD are Riemann-Liouville and Caputo fractional derivatives, respectively. Some of the properties of the Riemann-Liouville fractional operator are recalled below [Podlubny, 1999 a,b, Xu and Tan, 2006]:
***Property* (2.4.1):**

$$_{t_0}D_t^p (t-t_0)^v = \frac{\Gamma(1+v)}{\Gamma(1+v-p)} (t-t_0)^{v-p}$$

where $p \in R$ and $v > -1$.
***Property* (2.4.2)**:

$$_0D_t^p H(t) = \frac{t^{-p}}{\Gamma(1-p)}$$

where H(t) is the Heaviside unit step function.

Property **(2.4.3)**:

$${}_{t_0}D_t^p\left({}_{t_0}D_t^q f(t)\right) = {}_{t_0}D_t^{p+q} f(t) - \sum_{j=1}^{n}\left[{}_{t_0}D_t^{q-j} f(t)\right]_{t=t_0}\frac{(t-t_0)^{-p-j}}{\Gamma(1-p-j)}$$

where $p, q \in R$, $n \in Z$, and , $n-1 \le q < n$.

Property **(2.4.4)**:

$${}_0D_t^\gamma\left(t^{k\alpha+\beta-1}E_{\alpha,\beta}^{(k)}(\lambda t^\alpha)\right) = t^{k\alpha+\beta-\gamma-1}E_{\alpha,\beta-\gamma}^{(k)}(\lambda t^\alpha)$$

where $\gamma \in R$, $k \in Z \setminus Z^-$ and $E^{(k)}(y) = \frac{d^k}{dt^k}\, E(y)$.

Definition 2.5[15]. A function f on 0 ≤ t <∞ is said to be exponentially bounded if it satisfies an inequality of the form
$\|f(x,t)\| \le \varepsilon M e^{\sigma t}$
for some real constants M >0 and c, for all sufficiently large t.

3. Dimensionless Dynamical Systems

Let us consider the master chaotic system in the form of:

$$\frac{d^\alpha x_m}{dt^\alpha} = y_m$$
$$\frac{d^\alpha y_m}{dt^\alpha} = z_m \qquad (11)$$
$$\frac{d^\alpha z_m}{dt^\alpha} = a(-x_m - y_m - z_m + f(x_m))$$

and the slave chaotic system is assumed to represented by:

$$\frac{d^\alpha x_s}{dt^\alpha} = y_s$$
$$\frac{d^\alpha y_s}{dt^\alpha} = z_s \qquad (12)$$
$$\frac{d^\alpha z_s}{dt^\alpha} = a(-x_s - y_s - z_s + f(x_s))$$

where 0 <*a*< 1 is a numerical parameter for the both master and slave systems $f(x_m)$ and $f(x_s)$.

The synchronization errors for each state variables for the systems are defined as:

$$e_x = x_m - x_s$$
$$e_y = y_m - y_s \qquad (13)$$
$$e_z = z_m - z_s$$

and the error dynamical system is expressed as:

$$\frac{d^\alpha e_x}{dt^\alpha} = e_y$$
$$\frac{d^\alpha e_y}{dt^\alpha} = e_z \qquad (14)$$
$$\frac{d^\alpha e_z}{dt^\alpha} = -a(e_x + e_y + e_z - f(x_m) - f(x_s))$$

3.4 . Fractional Bifurcation

Consider the system:

$$\frac{d^\alpha y}{dt^\alpha} = Ay + f(y(t),t) \qquad (15)$$

where A is real matrix with the characteristic roots all having negative real parts, f is real-valued continuous function. For small $\|y\|$ and $t \geq 0$,and$\|f(y,t)\| \leq \varepsilon\|y\|$

Proof:
According to El-Borai [16], we can write:

$$y(t) = \int_0^\infty y(0)\xi_\alpha(\theta)e^{A\theta t^\alpha}d\theta +$$

$$\alpha\int_0^t\int_0^\infty \theta\xi_\alpha(\theta)(t-s)^{\alpha-1}e^{A\theta(t-s)^\alpha}f d\theta\, ds$$

since the real parts of the characteristic roots of A are negative, there exists a positive constants λ such that:

$\left\|e^{A\theta t^\alpha}\right\| \leq e^{-\lambda\theta t^\alpha}$

Then

$$\|y(t)\| \leq \int_0^\infty \|y(0)\|\xi_\alpha(\theta)e^{-\lambda\theta t^\alpha}d\theta +$$

$$\alpha\int_0^t\int_0^\infty \theta\xi_\alpha(\theta)(t-s)^{\alpha-1}e^{-\lambda\theta(t-s)^\alpha}\|f\|d\theta\, ds \qquad (16)$$

$$\|y(t)\| \leq \int_0^\infty \|y(0)\|\xi_\alpha(\theta)e^{-\lambda\theta t^\alpha}d\theta +$$

$$\alpha\varepsilon\int_0^t\int_0^\infty \theta\xi_\alpha(\theta)(t-s)^{\alpha-1}e^{-\lambda\theta(t-s)^\alpha}\|y(s)\|d\theta\, ds$$

Where $\int_0^\infty \theta\xi_\alpha(\theta)e^{-\lambda\theta(t-s)^\alpha}d\theta = M$

$$\|y(t)\| \leq \|y(0)\|E_\alpha(-\lambda t^\alpha) + \varepsilon M\int_0^t (t-s)^{\alpha-1}\|y(s)\|ds$$

$$\|y(t)\| \leq \|y(0)\| + \varepsilon M\int_0^t (t-s)^{\alpha-1}\|y(s)\|ds(17)$$

Where $y(t) = e^{\sigma t}x(t)$,

$$e^{\sigma t}\|x(t)\| \leq \|x(0)\| + \varepsilon M\int_0^t (t-s)^{\alpha-1}e^{\sigma t}\|x(s)\|ds$$

The last inequality yields:

$$e^{\sigma t}\|x(t)\| \leq e^{Mt\varepsilon}\|x(0)\|$$

or

$$\|x(t)\| \leq e^{(M\varepsilon-\sigma)t}\|x(0)\| \qquad (18)$$

For sufficiently small $\varepsilon > 0$,we get:

$\lim_{t\to\infty} \|x(0)\| = 0$

5. Fractional Differential Equations

We are concerned with the fractional nonlinear differential equation of the form: ${}^{c}D^{\alpha}x = Ax(t) + f(x(t),t)$, with the initial condition:

$x(0) = x_0$ (19)

$where \quad 0 < \alpha \leq 1$

and the differentiation ${}^{c}D^{\alpha}x$ is of Caputo type. We assume that $f: [0,\infty) \times R \to R$ is continuous. Under the continuity condition, one can show that (19) is equivalent to the Volterra fractional integral equation:

$$I^{\alpha}\frac{d^{\alpha}x}{dt^{\alpha}} = x(t) - x(0) \qquad (20)$$

$$I^{\alpha}[Ax + f] = x(t) - x(0)$$

$$\big|\|x(t)\| - \|x(0)\|\big| \leq \|x(t) - x(0)\| = \|I^{\alpha}[Ax + f]\|$$

$$\leq \|I^{\alpha}Ax\| + \|I^{\alpha}f\| \leq \frac{1}{\Gamma(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1}Ax(s)ds\right\| + \|I^{\alpha}f\|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|Ax(s)\|ds + M\|I^{\alpha}x\|$$

$$\|x(t)\| - \|x(0)\| \leq M\|I^{\alpha}x\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|Ax(s)\|ds$$

$$\|x(t)\| \leq \|x(0)\| + M\|I^{\alpha}x\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|Ax(s)\|ds \qquad (21)$$

$x(0) = x_0 \qquad t \geq 0$

Where Γ is the gamma function.Equation (21) have a wide range of applications to the real-world problems. We observe that the equation (21) is a singular integral equation with a convex kernel.The classical example of the equation (19) is:

$${}^{c}D^{\alpha}x = Ax(t) + f(x(t),t) \qquad (22)$$

$x(0) = x_0 \;, \qquad 0 < \alpha < 1$

Where A is the real matrix with the characteristic roots all having negative real parts, $f: [0,\infty) \times R \to R$ is continuous. Using the solution of equation (22) is given by:

$$x(t) = \int_0^{\infty} x(0)\xi_{\alpha}(\theta)e^{A\theta t^{\alpha}}d\theta +$$

$$\alpha \int_0^t \int_0^\infty \theta \xi_\alpha(\theta)(t-s)^{\alpha-1} e^{A\theta(t-s)^\alpha} f d\theta\, ds$$

where$\|e^{A\theta t^\alpha}\| \le e^{-\lambda\theta t^\alpha}$

$$\|x(t)\| \le \int_0^\infty \|x(0)\| \xi_\alpha(\theta) e^{-\lambda\theta t^\alpha} d\theta +$$

$$\alpha \int_0^t \int_0^\infty \theta \xi_\alpha(\theta)(t-s)^{\alpha-1} e^{-\lambda\theta(t-s)^\alpha} \|f\| d\theta\, ds$$

$$\|x(t)\| \le \int_0^\infty \|x(0)\| \xi_\alpha(\theta) e^{-\lambda\theta t^\alpha} d\theta +$$

$$\alpha\varepsilon \int_0^t \int_0^\infty \theta \xi_\alpha(\theta)(t-s)^{\alpha-1} e^{-\lambda\theta(t-s)^\alpha} \|x(s)\| d\theta\, ds$$

Where:

$$\int_0^\infty \xi_\alpha(\theta) e^{-\lambda\theta t^\alpha} d\theta = E_\alpha(-\lambda t^\alpha)$$

and

$$\int_0^\infty \theta \xi_\alpha(\theta) e^{-\lambda\theta(t-s)^\alpha} d\theta = E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)$$

$$\|x(t)\| \le \|x(0)\| E_\alpha(-\lambda t^\alpha) + \alpha\varepsilon \int_0^t E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)(t-s)^{\alpha-1} \|x(s)\| ds$$

WhereE_α isMittag-Leffler functions, $E_{\alpha,\ \alpha}$ is Mittag-Leffler functions with two parameters [17].

Note that the solution of the ordinary differential equation:

$$\frac{dx}{dt} = Ax + f(x(t),t) \quad , \quad x(0) = x_0 \qquad (23)$$

can be expressed as:

$$\|x(t)\| \le \|x(0)\| e^{-\lambda t} + \varepsilon \int_0^t e^{-\lambda(t-s)} \|x(s)\| ds$$

In the following, we discuss the Laplace transform of the fractionalmethod .

6. Laplace transform of the fractional

We now consider the equation

$$^cD^\alpha x = Ax(t) + f(x(t),t) \qquad (24)$$

$$x(0) = x_0 \quad , \qquad 0 < \alpha < 1 \qquad t \ge 0$$

6.1. Laplace transform of the fractional integral[18].

$$_0I_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt \qquad (25)$$

where$\Re(\alpha) > 0$.

Application of convolution theorem of the Laplace transform gives

$$L\{ {}_0I_x^{-\alpha} f(x); s\} = L\left\{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right\} L\{f(t); s\} = s^{-\alpha}F(s) \qquad (26)$$

where $\Re(s) > 0$, $\Re(\alpha) > 0$.

6.2. Laplace transform of the fractional derivative[19]

If $n \in \mathbb{N}$, then by the theory of the Laplace transform, we know that

$$\left\{\frac{d^n}{dx^n} f; s\right\} = s^n\, F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0_+) (27)$$

$$= s^n\, F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0_+) \quad ,(n-1 \le \alpha < n)$$

where $\Re(s) > 0$ and $F(s)$ is the Laplace transform of $f(t)$.By virtue of the definition of the derivative, we find that

$$L\{ {}_0D_x^{\alpha} f; s\} = \boldsymbol{L}\left\{\frac{d^n}{dx^n}\ {}_0I_x^{n-\alpha} f; s\right\}$$

$$= s^n \boldsymbol{L}\{ {}_0I_x^{n-\alpha} f; s\} - \sum_{k=0}^{n-1} s^k \frac{d^{n-k-1}}{dx^{n-k-1}}\ {}_0I_x^{n-\alpha} f(0_+)$$

$$= s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^k\, D^{\alpha-k-1} f(0_+), (D = \frac{d}{dx})$$

$$= s^{\alpha} F(s) - \sum_{k=1}^{n} s^{k-1}\, D^{\alpha-k} f(0_+) (28)$$

where $\Re(s) > 0$.

6.3. Laplace transform of Caputo derivative

Definition 6.3.1

The Caputo derivative of a casual function f (t) (that isf (t) = 0 for t <0) with α> 0 was defined by Caputo (1969) in the form

$$C_0\ {}_aD_x^{\alpha} f(x) = {}_aI_x^{n-\alpha} \frac{d^n}{dx^n} f(x)$$

$$= {}_aD_t^{-(n-\alpha)} f^n(x) (3.1.12)$$

$$= \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^n(t) dt \quad (n-1 < \alpha < n) (29)$$

where $n \epsilon \mathbb{N}$. From (25) and (29), it follows that

$$\boldsymbol{L}\{C_0\ {}_aD_x^{\alpha} f(s); s\} = s^{-(n-\alpha)}\ \boldsymbol{L}\{f^n(t)\}$$

On using (27), we see that

$$L\{C_0\ {}_aD_x^{\alpha} f(s); s\} = s^{-(n-\alpha)} \left[s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^k(0_+) \right]$$

$$= s^{\alpha} - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{k}(0_{+}) (n-1 < \alpha < n) \quad (30)$$

where $\Re(s) > 0$ and $\Re(\alpha) > 0$.

Theorem 6.1. Assume (24) has a unique continuous solution $x(t)$, if $f(x(t),t)$ is continuous on $[0,\infty)$ and exponentially bounded,then $x(t)$, and its Caputo derivative ${}_0D_t^{\alpha}x(t)$are both exponentially bounded, thus their Laplace transforms exist.

Proof. Since $f(x(t),t)$ is exponentially bounded, there exist positive constantsM, σ and enough large T such that$\|f(x,t)\| \leq \varepsilon M e^{\sigma t}$for all t ≥ T .It is easy to see that Eq. (24) is equivalent to the following integral equation

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (Ax(\tau) + f(x(\tau),\tau)) d\tau$$

$$0 \leq t < \infty \quad (31)$$

For t≥ T ,(31) can be rewritten as

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^T (t-\tau)^{\alpha-1} (Ax(\tau) + f(x(\tau),\tau)) d\tau$$

$$+ \frac{1}{\Gamma(\alpha)} \int_T^t (t-\tau)^{\alpha-1} (Ax(\tau) + f(x(\tau),\tau)) d\tau$$

In view of assumptions of Theorem3.1,the solution $x(t)$, $x(0) = x_0$ is unique and continuous on [0,∞), then $(Ax(\tau) + f(x(\tau),\tau))$ is bounded on [0, T], i.e., there exists a constant K >0 such that$\|(Ax(\tau) + f(x(\tau),\tau))\| \leq$ K. We have

$$\|x(t)\| \leq \|x(0)\| + \frac{1}{\Gamma(\alpha)} K \int_0^T (t-\tau)^{\alpha-1} d\tau$$

$$+ \frac{1}{\Gamma(\alpha)} \int_T^t (t-\tau)^{\alpha-1} \|A\| \|x(\tau)\| d\tau$$

$$+ \frac{1}{\Gamma(\alpha)} \int_T^t (t-\tau)^{\alpha-1} \|f(x,t)\| d\tau$$

Multiply this inequality by$e^{-\sigma t}$and note that$e^{-\sigma t} \leq e^{-\sigma T}$, $e^{-\sigma t} \leq e^{-\sigma \tau}$, $\|f(x,t)\| \leq \varepsilon M e^{\sigma t}$(t ≥ T) to obtain

$$e^{-\sigma t}\|x(t)\| \leq e^{-\sigma t}\|x(0)\| + \frac{e^{-\sigma t}}{\Gamma(\alpha)} K \int_0^T (t-\tau)^{\alpha-1} d\tau$$

$$+\frac{e^{-\sigma t}}{\Gamma(\alpha)}\int_{T}^{t}(t-\tau)^{\alpha-1}\|A\|\|x(\tau)\|d\tau+\frac{e^{-\sigma t}}{\Gamma(\alpha)}\int_{T}^{t}(t-\tau)^{\alpha-1}\|f(x,t)\|d\tau$$

$$\leq e^{-\sigma t}\|x(0)\|+\frac{e^{-\sigma T}}{\alpha\Gamma(\alpha)}K(t^{\alpha}-(t-T)^{\alpha})$$

$$+\frac{\|A\|}{\Gamma(\alpha)}\int_{T}^{t}(t-\tau)^{\alpha-1}\|x(\tau)\|e^{-\sigma\tau}d\tau$$

$$+\frac{1}{\Gamma(\alpha)}\int_{T}^{t}(t-\tau)^{\alpha-1}\|f(x,t)\|e^{-\sigma\tau}d\tau$$

$$\leq e^{-\sigma t}\|x(0)\|+\frac{e^{-\sigma T}}{\alpha\Gamma(\alpha)}K(t^{\alpha}-(t-T)^{\alpha})$$

$$+\frac{\|A\|}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}\|x(\tau)\|e^{-\sigma\tau}d\tau+\frac{\varepsilon M}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}e^{\sigma(\tau-t)}d\tau$$

$$\leq e^{-\sigma t}\|x(0)\|+\frac{T^{\alpha}e^{-\sigma T}K}{\alpha\Gamma(\alpha)}$$

$$+\frac{\|A\|}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}\|x(\tau)\|e^{-\sigma\tau}d\tau+\frac{\varepsilon M}{\Gamma(\alpha)}\int_{0}^{t}s^{\alpha-1}e^{-\sigma s}d\tau$$

$$\leq e^{-\sigma t}\|x(0)\|+\frac{T^{\alpha}e^{-\sigma T}K}{\alpha\Gamma(\alpha)}+\frac{\|A\|}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}\|x(\tau)\|e^{-\sigma\tau}d\tau+\frac{\varepsilon M}{\sigma^{\alpha}}$$

Denote

$$a=e^{-\sigma t}\|x(0)\|+\frac{T^{\alpha}e^{-\sigma T}K}{\alpha\Gamma(\alpha)}+\frac{\varepsilon M}{\sigma^{\alpha}},\ b=\frac{\|A\|}{\Gamma(\alpha)},\ r(t)=\|x(\tau)\|e^{-\sigma\tau}$$

we get

$$r(t)\leq a+b\int_{0}^{t}(t-\tau)^{\alpha-1}r(\tau)\,d\tau$$

By Lemma 2.1,

$$r(t)\leq a\,E_{\alpha}(\theta t)=\sum_{k=0}^{\infty}\frac{(b\Gamma(\alpha))^{k}t^{k\alpha}}{\Gamma(k\alpha+1)}\quad t\geq T$$

Recall that the Mittag-Leffler function is defined as

$$E_{\alpha}(t)=\sum_{k=0}^{\infty}\frac{t^{k}}{\Gamma(k\alpha+1)}$$

Then

$$r(t)\leq aE_{\alpha}(b\Gamma(\alpha)t^{\alpha})\qquad t\geq T\qquad (31)$$

It is known that (see [20], p. 21) the Mittag-Leffler type function $E_{\alpha}(\omega t^{\alpha})$ satisfies the following inequality

$$E_{\alpha}(\omega t^{\alpha})\leq Ce^{\omega^{\frac{1}{\alpha}}t}\quad t\geq 0,\ \omega\geq 0,\ 0<\alpha<2,\qquad (32)$$

where C is a positive constant.With (31) and (32), we have

$$r(t) \le \ a\,C\,e^{(b\Gamma(\alpha))^{\frac{1}{\alpha}}t} \qquad t \ge T$$

Then

$$\|x(t)\| \le \ a\,C\,e^{((b\Gamma(\alpha))^{\frac{1}{\alpha}}+\sigma)t} t \ge T$$

From Eq. (24), we obtain

$$\|\,{}_0D_t^{\alpha}x(t)\| \le \|A\|\|x(t)\| + \|f(x(t),t)\| \le \|A\|a\,C\,e^{((b\Gamma(\alpha))^{\frac{1}{\alpha}}+\sigma)t} + \varepsilon M e^{\sigma t}$$

$$\le (\|A\|a\,C + \varepsilon M)e^{((b\Gamma(\alpha))^{\frac{1}{\alpha}}+\sigma)t} t \ge T$$

Taking Laplace transform with respect to t in both sides of (24), we obtain

$$x(s) = s^{\alpha-1}(s^{\alpha}-A)^{-1}x(0) + (s^{\alpha}-A)^{-1}f(x(s),s) \quad (33)$$

Taking The inverse Laplace transform (33) we obtain

$$x(t) = x(0)E_{\alpha}(At^{\alpha}) + \int_0^t E_{\alpha,\alpha}(A(t-s)^{\alpha})(t-s)^{\alpha-1}\,x(\tau)d\tau \qquad (34)$$

The Laplace transform method is suitable for constant coefficient fractional differential equations, but it demands for forcing terms, so not every constant coefficient fractional differential equation can be solved by the Laplace transform method.

Bibliography

[1] McCann, K. and Yodzis, P. (1995). Bifurcation structure of a three-species food chain model.Theoretical Population Biology, 48:93–125.
[2] Kuznetsov, Y. A., Muratori, S., and Rinaldi, S. (1992). Bifurcations and chaos in a periodic predator-prey model. International Journal of Bifurcation and Chaos., 2:117–128.
[3] Bazykin, A. D. (1998). Nonlinear Dynamics of Interacting Populations.World Scientific, Singa-pore.
[4]Boer, M. P. (2000). The dynamics of tritrophic food chains.PhD thesis, Free University, Ams-terdam.
[5] Wang, G. Perez, and H. A. Cerdeira.Dynamical behavior of the firings in acoupled neuronal system. *Phys. Rev.* E, 47(4):2893–2898, 1993
[6]Beyn,W.-J. [1991],Numerical methods for dynamical systems, *in* W. Light,ed.,‘Advances in Numerical Analysis,Vol. I,Nonlinear Partial Differential Equations and Dynamical Systems’,Oxford University Press,Oxford,pp. 175–236.
[7]Kuznetsov, Y. A. (2004). Elements of Applied Bifurcation Theory., volume 112 of Applied Mathematical Sciences.Springer, 3th edition.
[8]Beyn,W.-J. [1994],‘Numerical analysis of homoclinic orbits emanating from a Takens-Bogdanov point’, *IMA J. Numer. Anal.* **14**,381–410.
[9] Kuznetsov,Y. [1997], Explicit normal form corfficients for all codim 2 bifurcations of equilibria in ODEs ,Centrum voorWiskunde en Informatica,
Amsterdam. Report MAS-R9730.
[10] S. D. Eyldelman, Parabolic System, North- MollandPublisghing Company Amsterland, (1969).

[11] Gary M. Lieberman. Second order parabolic differential equations. World Scientific Publishing Co . Inc, River Edge , NJ, 1996.
[12] L. M. Pecora and T. L. Carroll, "Synchronization in Chaotic Systems," Physical Review Letters, Vol. 64, No. 8, pp. 821-824. doi:10.1103/PhysRevLett.64.821, (1990).
[13] M.M.A.Metwali, On solutions of quadretic integral equations,Adam Mickiewicz University. (2013).
[14] "Synchroniztion in Complex Networks" Physics Reports, 469 (3):93-153-doi:10.1061.j. Physrep. 09.002, (2008).
[15]Newhouse,S.,Palis,J. &Takens,F. [1983],'Bifurcations and stabilityof families of diffeomorphisms', *Inst. Hautes ´ Etudes Sci. Publ. Math.***57**,5–71.
[16] Mahmoud M. El-Borai, Wagdy G. Elsayed, TurkiyaAlhadi, Aljamal 'Synchronization and Impulsive Control of Some Parabolic Partial DifferentialEquations, American Journal of Theoretical and Applied Statistics, 2017; 6(5-1): 30-39
[17] R. Z. Luo, "Impulsive Control and Synchronization of a New Chaotic System," Physics Letters A, Vol. 372, No. 5, (2008) pp. 648-653. doi:10.1016/j.physleta.2007.08.010.
[18] Liu XZ, Willms AR. Impulsive controllability of linear dynamical systems with applications to maneuvers of spacecraft. MPE 1996;2:277–99.

[19] Mahmoud M. El- Borai, Wagdy G. Elsayed and Ragab M. Al- Masroub, Exact solutions of some nonlinear complex fractional partial differential equations, international Journal of Mathematics Trends and Technology. (IJMTT), April 2016, vol. 32, No. 1, 4-9.

[20] Mahmoud M. El- Borai, Wagdy G. Elsayed and Ragab M. Al- Masroub, Exact solutions for time fractional coupled Witham-Boer -Kaup equations via exp- function method, international research Journal of Engineering and Technology (IRJET), September 2015, vol. 2, Issue 6, 307-315.
[21] LeVeque RJ. Finite volume methods for hyperbolic problems .Cambridge University Press; 2002 .
[22] Yang T, Yang CM, Yang LB. Control of Ro¨ ssler system to periodic motions using impulsive control methods. PhysLett A 1997;232:356–61.
[23] Yang T, Yang LB, Yang CM. Impulsive control of Lorenz system. Phys D 1997;110:18–24 .
[24] Mahmoud M. El- Borai, Partial differential equations in an unbounded medium , Academia Nazionalle Dei Lincei Series VIII vol. IV December 6, 1973, 657-622.
[25] Yang T, Suykens JA, Chua LO. Impulsive control of non-autonomous chaotic systems using practical stabilization.Int J BifurcatChaos 1998;8(7):1557–64.
[26] Mahmoud M. El- Borai, Partial differential equations in an unbounded medium, Academia Nazionalle Dei Lincei Series VIII vol. IV December 6, 1973, 657-622.
[27] ErbeLH , Freedman HI, Liu XZ, Wu JH. Comparison principles for impulsive parabolic equations with applications to models of single species growth. J Austral Math SocSer B 1991;32:382–400.
[28] Stojanovski T, Kocarev L, Parlitz U. Driving and synch-ronizing by chaotic impulses. Phys Rev E 1996;43(9):782–5.
[29] Yang T, Chua LO. Impulsive stabilization for control and synchronization of chaotic systems: theory and application to secure communication. IEEE Trans Circuits System I 1997;44(10):976–88.
[30] Yang T, Chua LO. Impulsive control and synchronization of non-linear dynamical systems and application to secure communication.Int J Bifurcat Chaos 1997;7(3):645–64.

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