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# nجلةالتربوكا مجلة علمية محكمة تمار عنَكليةالتربية جامعة المرقبا 

العدد التاسع عشر
يوليو 2021م

## هيئـــة تحريـر

## هجلة التربوي

| اللمجلة ترحب بما يرد عليها من أبحاث وعلى استتداد لنشر ها بعد التحكيم . المجلة تحترم كل الاحترام آراء المحكمين وتعمل بمقتضاها كافة الآراء والأفكار المنشورة تعبر عن آراء أصحابها ولا تتحمل المجلة تبعاتها يتحمل الباحث مسؤولية الأماتة العلمية وهو المسؤول عما ينشر له . -البحوث المقدمة للنشر لا ترد لأصحابها نشرت أو لم تنشر (حقوق الطبع محفوظة للكلية) |
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ضو ابط النشر :<br>يشترط في البحوث العلمية المقدمة للنشر أن ير اعى فيها ما يأني :<br>- أصول البحث العلمي وڤو اعده<br>- ألا تكون المادة العلمية قد سبق نشر ها أو كانت جز عا من رسالة علمية . - برفق بالبحث تز كية لغوية وفق أنموذج معد<br>- تعدل البحوث المقبولة وتصحح وفق ما ير اه المحكمون . - النز ام الباحث بالضو ابط الني وضعتهها المجلة من عدد الصفحات ، ونوع الخط ورقمه ، و الفتزات الزمنية الممنوحة للتعديل ، ومـا يستجد من ضو ابط تضعها المجلة مستقبلا . تثبيهات :

- للمجلة الحق في تعديل البحث أو طلب تعديله أو رفضد . - يخضـع البحث في النشر لأولويات المجلة وسياستها . - البحوث المنشورة تعبر عن وجهة نظر أصحابها ، ولا تعبر عن وجهة نظر المجلة .


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## Hilbert Space and Applications

Elhadi A. A. Maree ${ }^{1}$, Abdualah Ibrahim Sultan ${ }^{2}$ and Khaled A. Alurrfi ${ }^{3}$<br>${ }^{1,3}$ Department of Mathematics \Faculty of Science - Elmergib University<br>${ }^{2}$ Department of Mathematics \Asmarya University Wisammaree11@gmail.com ${ }^{1}$, Sultan.1120006@gmail.com ${ }^{2}$, alurrfi@yahoo.com ${ }^{3}$


#### Abstract

Functional analysis is one of the most important branches of modern Mathematics and Physics. In functional analysis, Hilbert spaces have a great and positive influence in the development of functional analysis. Hilbert spaces have a rich geometric structure because they are endowed with a scalar product which allows the introduction of the concept of orthogonality of vectors. The aim of this paper is to introduce the Hilbert spaces, and their properties, operations, and applications. We first give an introduction about functional analysis and highlight the importance of Hilbert spaces in the development of functional analysis. Then, we focus on significant and crucial spaces for Hilbert spaces called Banach Spaces. We do so by studying normed spaces and their properties. Finally, we discuss what is meant by Hilbert spaces and what is the relation between Hilbert and Banach spaces.


## Keywords: Functional analysis, Hilbert Space, Banach Spaces.

## 1. Introduction

Mathematics is the queen of science and the language of nature. Also, it occupies and plays a crucial and important role in the human societies. Moreover, it represents a strategic key in the development of the whole world and mankind. It has roots in ancient Egypt and Babylonia, then grew rapidly in ancient Greece. Mathematics written in ancient Greek was translated into Arabic. About the same time some mathematics of India was translated into Arabic. Later some of this mathematics was translated into Latin and became the mathematics of Western Europe and then after a hundred years it became the mathematics of the whole world. In fact, Mathematics can be found in plenty of fields today for instance Business Decisions, Physics, Economics, Engineering, Medicine, Modern Society, Industry, Finance, Marketing and Computer Science. As we see nowadays, we are in the era of computer and technology, but if we think deeply for a moment, we will find that Mathematics has played a main and significant role in the high development of Computer Science. All the latest software and programmes that made the world very small and provided comfortable life for us has been connected with some Mathematical programmes.

There are plenty of branches of Mathematics such as Geometry, Topology, Algebra, Number Theory, Mathematical Analysis including Calculus and Real Analysis, Complex Variable Analysis, Differential Equations, Numerical Analysis and Functional Analysis etc. All these branches are relevant and connected with one another. And all of them have their own effects, significances, and applications in several fields. One of the most crucial and important branches is the Functional Analysis. Functional Analysis was born in the early years of the twentieth century as part of a larger trend toward abstraction. One of the characteristics of functional analysis is the study of classes of functionals in particular, the dual space, the space of all continuous linear functionals on a vector space (see[1-8]). Functional Analysis is

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one of the central areas of modern mathematics and it has a strong influence on a great number of completely different fields inside and outside of mathematics such as systems engineering and atomic physics, or some topics in general Topology, Measure and Integration, Linear Algebra, Geometry, the theory of approximation or representation theory and theory of Banach and Hilbert spaces. These spaces have a great impact in the development of Functional Analysis because the whole theory of Functional Analysis deals with them. Banach space is known as complete normed linear space while the complete inner product space is called Hilbert space. The typical examples of Banach spaces are $\mathbb{R}^{n}$ and $\mathbb{C}^{n}$ under the following norm:
$\|u\|=\sqrt{\sum_{j=1}^{n}\left|u_{j}\right|^{2}}$
where $u=\left(u_{1}, u_{2}, \ldots, u_{n}\right) \in \mathbb{R}^{n}$ or $\mathbb{C}^{n}$.
and,
$C_{b}(\mathbb{R})=\left\{f \in L^{\infty}(\mathbb{R}): f\right.$ is bounded and continuous $\}, \quad\|f\|_{\infty}=\sup _{x \in R}|f(x)|$
The typical examples of Hilbert space are as follows:
The space $\mathbb{C}^{n}$, finite dimensional complex Euclidean space, is a Hilbert space with an inner product expressed by:
$\langle u, v\rangle=u \cdot v=u_{1} \overline{v_{1}}+u_{2} \overline{v_{2}}+\cdots+u_{n} \overline{v_{n}}$.
where:
$u=\left(u_{1}, u_{2}, \ldots, u_{n}\right) \in \mathbb{C}^{n}$ and $v=\left(v_{1}, v_{2}, \ldots, v_{n}\right) \in \mathbb{C}^{n}$.
The space $\mathbb{R}^{n}$ with the following an inner product
$\langle u, v\rangle=u . v=u_{1} v_{1}+u_{2} v_{2}+\cdots+u_{n} v_{n} \quad$ where $u, v \in \mathbb{R}^{n}$.
The mathematical concept of a Hilbert space named after David Hilbert who made a very positive and great influence in the development of Functional analysis. It extends the methods of vector algebra and calculus from two and three dimensions to finite and infinite number of dimensions.

The theory of Hilbert spaces is the core around which functional analysis has developed. Hilbert spaces have a rich geometric structure because they are endowed with an inner product which allows the introduction of the concept of orthogonality of vectors. Hilbert space is truly fundamental mathematical structure which appears in wide branches of pure and applied mathematics. For instance, quantum mechanics, integral equations, linear system of equations and operator theory.
Also, Hilbert space is an abstract notion of great power and beauty which has been central to the development of mathematical analysis and forms the backdrop for many applications of analysis to science and engineering due to the existence of an inner product that determines the norm.

## 2. Normed Vector Spaces

### 2.1 Basics about Linear algebra

In order to achieve our aim of discussing the operators and the application of a Hilbert space, it is really important to know some concepts, definitions and examples of some related subjects such as vector space, vector subspace, normed space, Banach space etc. All these results will provide assistance and guide us to the main point of our research.

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Definition 2.1.1 (Vector Space).A vector space X (called also linear space) is a set of elements (vectors) over the field F , with two operations:
(i) A mapping $(x, y) \rightarrow x+y$ from $X \times X$ into $X$ called addition
(ii) A mapping $(\lambda, x) \rightarrow \lambda \times$ from $F \times X$ into $X$ called multiplication by scalars, such that the following conditions are satisfied :

$$
\begin{gathered}
x+y=y+x \\
(x+y)+z=x+(y+z) . \\
\alpha(x+y)=\alpha x+\alpha y . \\
(\alpha+\beta) x=\alpha x+\beta x . \\
(\alpha \beta) x=\alpha(\beta x) . \\
x+0=x . \\
0 x=0 . \\
1 x=x .
\end{gathered}
$$

where $x, y, z \in X$ and $\alpha, \beta \in F$
Elements of $X$ will be called vectors If $F=\mathbb{R}$, then $X$ will be called a real vector space, and if $F=\mathbb{C}$, then $X$ will be called a complex vector space.
Definition 2.1.2 (Vector Subspace).A subset $W \subset V$ of a vector space is a vector subspace of $V$ if thefollowing two axioms are satisfied:
(i) If $v, w$ are vectors in $W$, then so is $v+w$
(ii) For any scalar $\lambda \in \mathbb{R}$,
if $w$ is any vector in $W$, then so is hw
Vector subspace is also called linear subspace.
Definition 2.1.3 (Linear Combination). Let $E$ be a vector space and let $x_{1}, x_{2}, \ldots \ldots, x_{k} \in E$. A vector $x \in E$ is called a linear combination of vectors $x_{1}, x_{2}, \ldots . ., x_{k}$ if there exist scalars $\alpha_{1}, \alpha_{2}, \ldots \ldots, \alpha_{k}$ such that
$x=\alpha_{1} x_{1}+\cdots+\alpha_{k} x_{k}$
Definition 2.1.4 (Span).Given a set of vectors $S=\left\{v_{i} \in V: i \in I\right\}$ for a vector space $V$ and indexing set $I$, the collection of all linear combinations of vectors in $S$ is called thespan of $S$ and denoted by $\operatorname{span}(S)$
It is obvious that $\operatorname{span}(S)$ is a subspace of $V$ for any set $S$.

Definition 2.1.5 (Linear Independence).
A finite collection of vectors $\left\{x_{1}, x_{2}, \ldots, x_{\mathrm{k}}\right\}$ is called linearly dependent if $\alpha_{1} x_{1}+\cdots+\alpha_{k} x_{k}=0$ only if
$\alpha_{1}=\alpha_{2}=\cdots=\alpha_{k}=0$.
A collection of vectors which is not linearly independent is called linearly dependent.

### 2.2 Basic concepts and definition

Definition 2.2.1 (Metric Space).Let $X$ be a set. A metric on $X$ is a function:

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$$
d: X \times X \rightarrow \mathbb{R}^{+}, \quad(x, y) \mapsto d(x, y)
$$

satisfying the following properties:
(M1) $d(x, y) \geq 0$, and $d(x, y)=0$ if and only if $x=y$
(M2) $d(x, y)=d(y, x)$ for all $x, y \in X$
(M3) $d(x, z) \leq d(x, y)+d(y, z)$ for all $x, y, z \in X \quad$ (triangle inequality)
A metric space ( $X, d$ ) is a non-empty set $X$ on which a metric $d$ is given.
Definition 2.2.2 (Norm).Assume $X$ be a vector space over $K-($ i.e. $\mathbb{R}$ or $\mathbb{C}$ ). A norm on $X$ is a map

$$
\|\cdot\|: \quad X \rightarrow[0, \infty[, \quad x \mapsto\|x\|
$$

Satisfying the following properties:

$$
\begin{aligned}
& \text { (S2) }\|\lambda x\|=|\lambda|\|x\| \text { for every } x \in X \text { and } \lambda \in R \\
& \text { (S1) }\|x\| \geq 0, \quad \text { and }\|x\|=0 \text { if and only if } x=0 \\
& \text { (S3) }\|x+y\| \leq\|x\|+\|y\| \quad \text { (triangle inequality) } \forall x, y \in X
\end{aligned}
$$

Remark: In case the function $\|$.$\| satisfies only the properties \left(S_{2}\right)$ and $\left(S_{3}\right)$, Then it is called a seminorm on
Definition 2.2.3 (Normed Space)A vector space with a norm is called normedspace.
Also we can say that a normedspace is a pair $(\boldsymbol{X},\|\|$.$) , where \mathrm{X}$ is a vector space and $\|$.$\| is$ a norm defined on X .

### 2.3 Properties of Normed Spaces

### 2.3.1 Sequences

Sequence. A Sequence in a normed space $X$ is an ordered set in $X$ whose members can be labelled with positive integers. We write $\left\{u_{1}, u_{2}, \ldots\right\}$ or $\left\{u_{k}\right\}_{k=1}^{\infty}$.
Convergence of sequences. A sequence $\left\{u_{n}\right\}$ in a subset $Y$ of a normed space $X$ isconvergent if there is a member $u \in Y$ for which, given any $\epsilon>0$, a number $N$ can be found such that

$$
\left\|u_{n}-u\right\|<\epsilon \quad \text { for all } n>N
$$

If this is the case, we write $u_{n} \rightarrow u$ (which is read ' $u_{n}$ converges to $u$ ') and $u$ is called the limit of the sequence. So, in other way we can express it by

$$
\lim _{n \rightarrow \infty}\left\|u_{n}-u\right\|=0 \quad \text { or } \quad \lim _{n \rightarrow \infty} u_{n}=u
$$

Which is read 'the limit as $n$ tends to $\infty$ of $\boldsymbol{u}_{\boldsymbol{n}}$, is $\boldsymbol{u}^{\prime}$.
The convergence in a normed space has the basic properties of the convergence in R .
(i)A convergent sequence has a unique limit
(ii) If $x_{n} \rightarrow x \lambda_{n} \rightarrow \lambda\left(\lambda, \lambda_{n}\right.$ are scalars $)$, then $\lambda_{n} x_{n} \rightarrow \lambda$
(iii) If $x_{n} \rightarrow x$ and $y_{n} \rightarrow y, \quad$ then $x_{n}+y_{n} \rightarrow x+y$

Remark. A norm in a vector space $\boldsymbol{E}$ induces a convergence in $\boldsymbol{E}$. In other words, if we have a normed space $\boldsymbol{E}$, then we automatically have a convergence defined in $\boldsymbol{E}$.
Uniform convergence: Let $f, f_{1}, f_{2}, \ldots, \in \boldsymbol{\ell}(\boldsymbol{A})$. We say that the sequence $\left\{f_{n}\right\}$ convergence uniformly to $f$ if for every $\epsilon>0$ there exist a constant $M$ such that for all $x \in A$ and for all indices $n \geq M$ we have

$$
\left|f(x)-f_{n}(x)\right|<\epsilon
$$

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Where $\boldsymbol{\ell}(\boldsymbol{A})$ is the space of all continuous functions defined on a closed bounded set $A \subset R^{n}$.
Pointwise convergence: Let. We say that the sequence $\left\{f_{n}\right\}$ is Pointwise convergence to $f$ if

$$
f_{n}(t) \rightarrow f(t) \text { for every } t \in[a, b]
$$

Where $\ell([0,1])$ is the space of all continuous functionsdefined on the interval $[0,1]$.

### 2.3.2 Completeness

Definition 2.3.2.1.A sequence $\left\{u_{n}\right\}$ in a subset $Y$ of a normed space $X$ is called a Cauchy sequence if

$$
\lim _{m, n \rightarrow \infty}\left\|u_{m}-u_{n}\right\|=0
$$

or, in other words, more formally, if for any given $\epsilon>0$ there exist a number $N$ such that
$\left\|u_{m}-u_{n}\right\|<\epsilon \quad$ whenever $m, n>N$
Definition 2.3.2.2.Let $X$ be a normed space and let $\left\{x_{n}\right\}_{n} \subset X$ be a sequence. Then ( $x_{n}$ ) is said to be bounded if there is a constant $c>0$ such that,

```
|xn|\leqc\quad\foralln\inN
```

Definition 2.3.2.3.A normed space $X$ is called complete if every Cauchy sequence in $X$ converges to an element of $X$.
Definition 2.3.2.4.A Branch Space is a normed space which is complete in the metric defined by its norm; this meant that every Cauchy sequence is required to converge.
Proposition 2.3.2.1. Let $X, Y$ be normed spaces and $\operatorname{let} Z \subset X$ be a vector subspace. Then, the space $X \times Y$ becomes with the norm

$$
\|(x, y)\|:=\left(\|x\|_{X}^{2}+\|y\|_{Y}^{2}\right)^{\frac{1}{2}}
$$

is a normed space.
Proof. Note that coordinate wise addition/multiplication.
$\alpha\left(x^{\prime}, y^{\prime}\right)+\beta(x, y):=\left(\alpha x^{\prime}+\beta x, \alpha y^{\prime}+\beta y\right)$
Makes $X \times Y$ a vector space ( $S_{1}$ ) and ( $\mathrm{S}_{2}$ ) ofa norm are obvious.
For $\left(\mathrm{S}_{3}\right)$ We consider $(x, y)$ and $\left(x^{\prime}, y^{\prime}\right) \in X \times Y$.Then

$$
\begin{gathered}
\left\|(x, y)+\left(x^{\prime}, y^{\prime}\right)\right\|=\left(\left\|x+x^{\prime}\right\|_{X}^{2}+\left\|y+y^{\prime}\right\|_{Y}^{2}\right)^{\frac{1}{2}} \\
\leq\left[\left(\|x\|_{X}+\left\|x^{\prime}\right\|_{X}\right)^{2}+\left(\|y\|_{Y}+\left\|y^{\prime}\right\|_{Y}\right)^{2}\right]^{\frac{1}{2}} \\
=\left[\|x\|_{X}^{2}+\left\|x^{\prime}\right\|_{X}^{2}\right]+2\|x\|_{X}\left\|x^{\prime}\right\|_{X}+\|y\|_{Y}^{2}+\left\|y^{\prime}\right\|_{Y}^{2}+2\|y\|_{Y}\left\|y^{\prime}\right\|_{Y^{2}}^{\frac{1}{2}} \\
\leq\left[\|x\|_{X}^{2}+\|y\|_{Y}^{2}+2 \sqrt{\|x\|_{X}^{2}+\|y\|_{Y}^{2}} \cdot \sqrt{\left\|x^{\prime}\right\|_{X}^{2}+\left\|y^{\prime}\right\|_{Y}^{2}}+\left\|x^{\prime}\right\|_{X}^{2}+\left\|y^{\prime}\right\|_{Y}^{2}\right]^{\frac{1}{2}} \\
\left.=\left[\left(\|x\|_{X}^{2}+\|y\|_{Y}^{2}\right)^{\frac{1}{2}}+\left(\left\|x^{\prime}\right\|_{X}^{2}\right)+\left\|y^{\prime}\right\|_{Y}^{2}\right)^{\frac{1}{2}}\right]^{2 \cdot \frac{1}{2}} \\
=\|(x, y)\|+\left\|\left(x^{\prime}, y^{\prime}\right)\right\|
\end{gathered}
$$

Definition 2.3.2.5. Let X be a normed space and let $\left(x_{n}\right)_{n} \subset X$ be a sequence. Then,

$$
\text { (i) } x_{n} \text { converges to } x \leftrightarrow \lim _{n \rightarrow \infty}\left\|x_{n}-x\right\|=0
$$

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(ii) $\sum_{n=1}^{\infty} x_{n}$ converges if the sequence of partial sums $s_{N}$

$$
=\sum_{n=1}^{N} x_{n} \text { converges }
$$

Theorem 2.3.2.1.A closedvectorsubspaceofaBanach spaceisaBanach spaceitself.
Proof.Let $(E,\|\|$.$) be a Banach space and let F$ be a closed vector subspace of $E$. If $\left\{x_{n}\right\}$ is a Cauchy sequence in, then it is a Cauchy sequence in $E$ and therefore there exists $x \in E$ such that $x_{n} \rightarrow x$. Since $F$ is a closed subset of $E$, we have $x \in F$. Thus every Cauchy sequence in $F$ converges to an element of $F$.

Proposition 2.3.2.2.A Cauchy sequence is bounded.
Proof.Let $\left(x_{m}\right)_{m}$ be a Cauchy,

$$
\begin{gathered}
\exists N=N_{1}: \forall n, m \geq N:\left\|x_{n}-x_{m}\right\| \leq 1 . \text { Therefore, } \\
\left\|x_{n}\right\| \leq\left\|x_{n}-x_{N}\right\|+\left\|x_{N}\right\| \leq 1+\left\|x_{N}\right\| \quad \forall n \geq N
\end{gathered}
$$

So, we can find
$\left\|x_{m}\right\| \leq \max \left\{\left\|x_{1}\right\|,\left\|x_{2}\right\|, \ldots .,\left\|x_{N}\right\|,\left\|x_{N}\right\|+1\right\}<\infty \quad \forall m$.
Let Let $X$ be a metric space, $x \in X$ and $r>0$. We callDefinition 2.3.2.6.
$B_{r}(x)=\{y \in X: d(x, y)<r\}$ open ball
$K_{r}(x)=\{y \in X: d(x, y) \leq r\}$ closed ball
with centre $x$, radius $r>0$.
Proposition 2.3.2.3.Let $X, Y$ be complete metric spaces (respectively, Banach spaces) and $Z \subset X$ be a closed subset (respectively, a closed vector subspace).
(i) $X \times Y$ is complete metric space (Banach space)
(ii) $Z$ is a complete metric space (Banach space)

Theorem 2.3.2.2 (BaireTheorem). Let $X$ be a complete metric space and $\left(D_{j}\right)_{j \in \mathbb{N}}$ be countably many dense open subsets. Then $D:=\bigcap_{i \in \mathbb{N}} D_{i}$ is dense.

### 2.4 Linear Mappings

Definition 2.4.1 (Linear maps). Let $E$ and $F$ be $K$ - vector spaces. A linear mapA of $E$ into $F$ is a map $A: E \rightarrow F$ such thatLet $X, Y$ be normed spaces and let $A: X \rightarrow Y$ be a linear map. Then the following are equivalent:

$$
(\lambda x+\mu y)=\lambda A(x)+\mu A(y) \quad \text { for all } x, y \in E \text { and } \lambda, \mu \in K
$$

A linear map $A: E \rightarrow K$ is called a linear functionalor also, a linear form $E$.
The set of all linear forms on Eis called the algebraic dual $E^{*}$ of $E$.
$E^{*}$ becomes a $K$ - vector space with the following definitions of addition and scalar multiplication:

```
y+z:x\mapstoy(x)+z(x), \lambday:x\mapsto\lambday(x); y,z\in\mp@subsup{E}{}{*},\quad\lambda\inK,x\inE
```

The zero map 0:X $\rightarrow Y$ mapping every element $x \in X$ to $0 \in Y$.

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## Proposition 2.4.1.

(i)A linear map $A: X \rightarrow Y$ is called injectiveif for all $x_{1}, x_{2} \in X$, the condition $A x_{1}=A x_{2}$ implies that $x_{1}=x_{2}$. In other words, different vectors in $X$ are mapped to different vectors in $Y$.
(ii) A linear map $A: X \rightarrow Y$ is called surjectiveif the range of $A=Y$.
(iii) A linear map is called bijective if Ais injective and surjective.

Definition 2.4.2 (continuous mappings). Let $E_{1}$ and $E_{2}$ be two normed spaces, and let $L$ be a mapping from $E_{1}$ into $E_{2}$. If for any sequence $\left\{x_{n}\right\}$ of elements of $E_{1}$ converges to $x_{0} \in E_{1}$, the sequence $\left\{L\left(x_{n}\right)\right\}$ converges to $L\left(x_{0}\right)$, the mapping $L$ is called continuousat $x_{0}$, i.e., $L$ is continuous at $x_{0}$ if

$$
\begin{gathered}
\left\|x_{n}-x_{0}\right\| \rightarrow 0 \quad \text { implies } \quad\left\|L\left(x_{n}\right)-L\left(x_{0}\right)\right\| \rightarrow 0 \\
\text { or } x_{n} \rightarrow x_{0} \text { in } E_{1} \quad \text { implies } \\
L x_{n} \rightarrow L x_{0} \text { in } E_{2}
\end{gathered}
$$

If $L$ is continuous at every $x \in E_{1}$, then we simply say that $L$ is continuous.
We can also define the continuous mapping on a normed space by saying that:
Let $E$ be a normed space. A map $A: E \rightarrow E$ is said to be continuousif for $\epsilon>0$, there exist $\delta>0$ such that
$\|x-y\|<\delta \quad$ implies $\quad\|A(x)-A(y)\|<\epsilon \quad$ for all $x, y \in E$
Theorem 2.4.1. A linear mapping $L: E \rightarrow F$ is a continuous if and only if it is continuous at a point.
Proof. Assume $L$ is continuous at $x_{0} \in E$. Let $x$ be arbitrary element of $E$ and let $\left\{x_{n}\right\}$ be a sequence convergent to $x$. Then the sequence $\left\{x_{n}-x+x_{0}\right\}$ converges to $x_{0}$ and thus we have

$$
\left\|L\left(x_{n}\right)-L(x)\right\|=\left\|L\left(x_{n}-x+x_{0}\right)-L\left(x_{0}\right)\right\| \rightarrow 0
$$

## Definition 2.4.3 (Bounded linear mapping).

A linear mapping $L: E \rightarrow F$ is said to be bounded if there is a number $C$ such that
$\|L(x)\| \leq C\|x\| \quad$ for all $x \in E$
Proposition 2.4.2. Let $X, Y$ be normed spaces and let $A: X \rightarrow Y$ be a linear map. Then the following are equivalent:

> a) $A$ is continuous.
> b) $A$ is bounded. That is,
> $c:=\sup _{u \in X_{u \neq 0}} \frac{\|A u\|}{\|u\|}<\infty$
> Where $c=\sup _{u \neq 0} \frac{\left\|A_{u}\right\|}{\|u\|}$,
> $u \in X$ is the norm of the linear map $A$
> Proof. $\boldsymbol{b}) \Rightarrow \boldsymbol{a})$ If $c<\infty$, we get
> $\left\|A u_{n}-A u\right\|=\left\|A\left(u-u_{n}\right)\right\| \leq c .\left\|u-u_{n}\right\|$
> thus, $u_{n} \rightarrow u$ implies $A u_{n} \rightarrow A u$.

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$\boldsymbol{a}) \Rightarrow \boldsymbol{b}$ ) Assume $A$ were unbounded :c $\boldsymbol{c}=\infty$. Then

$$
\begin{gathered}
\forall n \in \mathbb{N} \quad \exists u_{n} \in X:\left\|A u_{n}\right\|>n \cdot\left\|u_{n}\right\| . \\
\text { Set } v_{n}:=\frac{u_{n}}{n\left\|u_{n}\right\|} \text { and observe that } \\
\left\|v_{n}\right\|=\frac{\left\|u_{n}\right\|}{n\left\|u_{n}\right\|}=\frac{1}{n} \rightarrow 0 . \\
\text { Thus } v_{n} \rightarrow 0 \text { and }\left\|A v_{n}\right\|=\left\|A \frac{u_{n}}{n\left\|u_{n}\right\|}\right\|=\frac{\left\|A u_{n}\right\|}{n\left\|u_{n}\right\|} \text { i.e. }\left\|A v_{n}\right\|>\frac{n\left\|u_{n}\right\|}{n\left\|u_{n}\right\|}=1 \\
\text { i.e. } A v_{n} \leftrightarrow A 0=0 .
\end{gathered}
$$

Definition 2.4.4. A mapping $A$ from a subset $E$ of a normed space $F$ into $F(A: F \rightarrow F)$ is called a contracting mapping(or simply acontraction) if there exist a positive number $c<1$ such that

$$
\|A u-A w\| \leq c\|u-w\| \quad \text { for every } u, w \in E
$$

## Theorem 2.4.2. (Banach Fixed Point / Contraction Mapping Principle).

Let $X$ be any complete metric space (Banach space) let $F \subset X$ be a closed subset and $A: F \rightarrow F$ be a contraction, i.e., a map satisfying

```
\| A u - A w \| \leq c \| u - w \| \quad \forall u , w \in F
```

Where $c$ is a constant belonging to the unit open interval $(0,1)$. Then, there is a unique fixed point $f \in F$, i.e., a point with $A f=f$.
Proof the uniqueness.Assume that $f, g$ are two fixed points with $f \neq g$. Then

$$
\begin{gathered}
A s f=A f \text { and } g=A g \text { from the uniqueness } \Rightarrow \\
0<\|f-g\|=\|A f-A g\| \\
\leq c\|f-g\|<\|f-g\| \text { contradiction ! since }\|f-g\| \text { can not be less } \\
\text { than }\|f-g\| \\
\Rightarrow\|f-g\|=0 \Rightarrow f=g
\end{gathered}
$$

So, there is just one unique fixed point.
Proof the existence.Write $A^{n} u$ for $A$ o $A$ o $A$ o ... ... o $A$ and $A^{o} u=u$.
We want to prove that $\left(A^{n} u\right)_{n}$ is Cauchy
One we get this; we conclude thatf $:=\lim _{\mathrm{n}} \mathrm{A}^{\mathrm{n}}$ uexists (by completeness) and it is a fixed point. Indeed.

$$
A f=A\left(\lim _{n} A^{n} u\right)=\lim _{n} A\left(A^{n} u\right)=\lim _{n} A^{n+1} u=f
$$

Now, to prove the sequence $\left(A^{n} u\right)_{n}$ is really Cauchy: take $n, k \in \mathbb{N}$. Then

$$
\begin{gathered}
\leq c^{n}\left\|u-A^{k} u\right\| \\
\leq c^{n}\left(\|u-A u\|+\left\|A u-A^{2} u\right\|+\cdots \ldots \ldots+\left\|A^{k-1} u-A^{k} u\right\|\right) \text { triangle inequality } \\
\leq c^{n}\left(\|u-A u\|+c\|u-A u\|+c^{2}\|u-A u\|+\cdots . .+c^{k-1}\|u-A u\|\right) \\
=c^{n}\left(1+c+c^{2}+\cdots+c^{k-1}\right)\|u-A u\|
\end{gathered}
$$

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$$
\begin{gathered}
\leq c^{n}\left(1+c+c^{2}+\cdots+c^{k-1}+\cdots\right)\|u-A u\| \\
=\frac{c^{n}}{1-c}\|u-A u\|
\end{gathered}
$$

Where we used the triangle inequality in the third line, the contraction property in the line 4 , and the geometric series in the last line.
Proposition 2.4.2. The function

$$
\left\lvert\,\|A\|\left\|:=\sup _{u \in X_{u \neq 0}} \frac{\|A u\|}{\|u\|}=\sup _{w \in X_{\|w\|=1}}\right\| A w\right. \|
$$

is a norm on $\mathcal{L}(X, Y)$.
where $\mathcal{L}(X, Y)$ is the space of all continuous linear maps from $X$ to $Y$ :
$\mathcal{L}(X, Y)=\{A: X \rightarrow Y:$ A linear, continuous $\}$.

## Theorem 2.4.3(Open Mapping Theorem).

Let $X, Y$ be Branch spaces, and let $A: X \rightarrow Y$ be a bounded, surjective linear map. Then $A$ maps open sets in $X$ to open sets in $Y$.

## Theorem 2.4.4(Inverse Mapping Theorem).

If $A$ is a bounded linear bijection from a Branch space $X$ to Branch space $Y$, then the inverse of $A$ is continuous (and hence bounded).

## Theorem 2.4.5(Closed Graph Theorem).

Let $A: X \rightarrow Y$ be a linear map between Branch spaces $X$ and $Y$. Then the graph $A$ is closed if and only if $A$ is bounded.

Given a map $A: X \rightarrow Y$, then the graph of $A$ is the set.

$$
G(A)=\{(x, y) \in X \times Y: y=A x\}
$$

Note that if $X$ and $Y$ are normed linear spaces, and $A$ is linear, then $G(A)$ is a linear subspace of $X \times Y$, with norm $\|x, y\|_{G(A)}=\|x\|_{X}+\|y\|_{Y^{*}}$
Proof. Suppose first that $A$ is bounded, and $\operatorname{let}\left\{\left(x_{n}, A x_{n}\right)\right\}$ be a sequence in $G(A)$ with limit $(x, y) \in X \times Y$. Since $A$ is bounded, then

$$
\left\|A\left(x_{n}-x\right)\right\|_{y} \leq\|A\|\left\|x_{n}-x\right\|_{X} \rightarrow 0, \quad \text { and so } A x_{n} \rightarrow A x .
$$

We also have

$$
\begin{gathered}
\left\|\left(x_{n}, A x_{n}\right)-(x, y)\right\|_{X \times Y} \rightarrow 0 \\
\Leftrightarrow\left\|x_{n}-x\right\|_{X}+\left\|A x_{n}-y\right\|_{Y} \rightarrow 0 \\
\Rightarrow A x_{n} \rightarrow y .
\end{gathered}
$$

Uniqueness of limits then implies that $y=A x$, and so $(x, y) \in G(A)$. Therefore $G(A)$ is closed.

Now suppose that $G(A)$ is closed. Then it is closed linear subspace of $X \times Y$, and in particular it is a Banach space.
We have two projection maps

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| $P_{1}: G(A) \rightarrow X$ $P_{2}: G(A) \rightarrow Y$ <br> $(x . A x) \mapsto x$ $(x, A x) \mapsto A x$ |  |  |

The following estimates show that $P_{1}$ and $P_{2}$ are bounded.

$$
\begin{aligned}
& \left\|P_{1}(x, A x)\right\|_{Y}=\|x\|_{X} \leq\|x\|_{X}+\|A x\|_{Y}=\|(x, A x)\|_{G(A)} \\
& \left\|P_{2}(x, A x)\right\|_{Y}=\|A x\|_{Y} \leq\|x\|_{X}+\|A x\|_{Y}=\|(x, A x)\|_{G(A)}
\end{aligned}
$$

Since $P_{1}$ is a bounded linear projection, then $P_{1}^{-1}$ is bounded by Theorem 6 (Inverse Mapping Theorem) (note that this uses the Open Mapping Theorem, and hence the Baire Category Theorem). Therefore $A=P_{2} \circ P_{1}^{-1}$ is bounded, which is complete the proof.

## 3. Hilbert Spaces

### 3.1 Some definitions and overview about Hilbert Spaces

In order to get some results that could provide help for the next sections, we need to define some spaces such as Inner Product Space and Hilbert Space and discuss some relations between Inner Product and Normed Space with some properties of Hilbert space. This section will be devoted for defining the required spaces.

Definition 3.1.1. Let $H$ be a vector space over the field $\mathbb{C}(K$ or $\mathbb{R})$. A function $(x, y) \rightarrow\langle x, y\rangle$ from $H \times H$ into $\mathbb{R}$ resp $\mathbb{C i s}$ called an inner product on $H$ if for all $x, y, z \in H$ and for all $\in \mathbb{R}$ or $\mathbb{C}$, we have

$$
\begin{gathered}
\text { (a) }\langle x, x\rangle \geq 0 \text {, and }\langle x, x\rangle=0 \\
\text { if and only if } x=0 \\
\text { (b) }\langle x+y, z\rangle=\langle x, z\rangle+\langle y, z\rangle \\
\text { (c) }\langle\lambda x, y\rangle=\lambda<x, y\rangle
\end{gathered}
$$

$$
\text { (d) }\langle x, y\rangle=\overline{\langle y, x\rangle}
$$

Aninner productis also called a scalar product.
Definition 3.1.2. A vector space equipped with a given inner product $\|x\|=\sqrt{\langle x, x\rangle}, \forall x \in H$ is called an inner product space (also called apre-Hilbert space).

The pair ( $H,<\ldots .>$ ) is called the inner product space.
Proposition 3.1.1. From the definition of inner product space, we obtained some results as following,

$$
\begin{gathered}
\text { (i) }<x, \lambda y>=\bar{\lambda}<x, y> \\
<x, \lambda y>=\overline{<\lambda y, x>}=\overline{\lambda<y, x>} \\
=\bar{\lambda}\langle y, x>=\bar{\lambda}<x, y>
\end{gathered}
$$

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$$
\begin{gathered}
\text { (ii) } \left.<\lambda_{1} x+\lambda_{2} y, z\right\rangle= \\
\left.\lambda_{1}<x, z\right\rangle+\lambda_{2}\langle y, z\rangle . \\
<\lambda_{1} x+\lambda_{2} y, z>=\overline{\left\langle z, \lambda_{1} x+\lambda_{2} y>\right.} \\
=\overline{\overline{\lambda_{1}}}\langle z, x\rangle+\overline{\overline{\lambda_{2}}}\langle z, y\rangle \\
\left.=\lambda_{1}\langle x, z\rangle+\lambda_{2}<y, z\right\rangle
\end{gathered}
$$

Proposition 3.1.2. Let $X$ be a pre- Hilbert space, $\mathbb{R}=K$, and $x, y \in X$. Then
(i) $\|x+y\|^{2}+\|x-y\|^{2}=2\|x\|^{2}+2\|y\|^{2}$

## (Parallelogram law)

(ii) $<x, y>=\frac{1}{4}\|x+y\|^{2}-\frac{1}{4}\|x-y\|^{2}$

## (Polarisation identity)

$$
\text { (iii) }<x, y>=0 \Rightarrow\|x+y\|^{2}=\|x\|^{2}+\|y\|^{2}
$$

## (Pythagoras)

Theorem 3.1.1.The inner product $(x, y) \rightarrow<x, y>$ from the product normed space $H \times H$ into $\mathbb{R}$ or $\mathbb{C}$ is continuous.
Proof. Suppose $x_{n} \rightarrow a$ and $y_{n} \rightarrow b$ in $H$. Then we have

$$
\begin{gathered}
\left|<x_{n}, y_{n}>-<a, b>\right|= \\
\left|<x_{n}, y_{n}>-<x_{n}, b>+<x_{n}, b>-<a, b>\right| \\
=\mid<x_{n}, y_{n}-b>-<x_{n}-a, b>1 \\
\leq \|<x_{n}, y_{n}-b>|-|<x_{n}-a, b>\mathrm{I} \\
\leq\left\|x_{n}\right\|\left\|y_{n}-b\right\|+\|b\|\left\|x_{n}-a\right\| \\
\text { When } x_{n} \rightarrow a \text { then }\left\|x_{n}-a\right\| \rightarrow 0 \text { as } n \rightarrow \infty \\
\text { since } y_{n} \rightarrow b \text { Moreover, }\left\|y_{n}-b\right\| \rightarrow 0 \text { as } n \rightarrow \infty \\
\text { Now, as }\left\|x_{n}\right\|\left\|y_{n}-b\right\|+\|b\|\left\|x_{n}-a\right\| \rightarrow 0 \\
\Rightarrow \| x_{n}, y_{n}>-<a, b>1 \rightarrow 0 \\
\text { And then, }<x_{n}, y_{n}>\rightarrow<a, b>
\end{gathered}
$$

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Definition 3.1.3. A Hilbert Space is a complete $\mathbb{R}$ or $\mathbb{C}$ - vector space equipped with a scalar (or inner) product.
From the definition we notice that a pre-Hilbert Spacewhich is complete under the norm $\|x\|=\sqrt{\langle x, x\rangle}$ is called a Hilbert Space.

### 3.2 Relation between norms and scalar product

All inner product spacesare also normed vector spaces. Given a scalar product $\|x\|:=\sqrt{\langle x, x\rangle} \forall x \in H$ is a norm.
Conversely, a scalar product comes from a norm only if the norm satisfies the parallelogram law.
Theorem 3.2.1.Every inner product space $H$ is a normed space under the norm

## Proof.

$\left(S_{1}\right)$ does hold, since $\sqrt{\langle x, x\rangle}=0$ when $\|x\|=0$. Also, the other way around is correct.
$\left(\boldsymbol{S}_{2}\right)$ does also hold, since

$$
\begin{gathered}
=\lambda \bar{\lambda}<x, x\rangle=|\lambda|^{2}\|x\|^{2} \\
\Rightarrow \quad\|\lambda x\|=|\lambda|\|x\| \quad \forall x \in H, \lambda \in \mathbb{R} \text { or } K
\end{gathered}
$$

$\left(S_{3}\right)$ In order to show the triangle inequality holds we consider

$$
\begin{gathered}
\|x+y\|^{2}=<x+y, x+y>= \\
<x, x>+<x, y>+<y, x>+<y, y> \\
=<x, x>+2 \operatorname{Re}<x, y>+<y, y> \\
\leq<x, x>+2|<x, y>|+<y, y> \\
\leq\|x\|^{2}+2\|x\|\|y\|+\|y\|^{2}=(\|x\|+\|y\|)^{2} \Rightarrow \\
\|x+y\| \leq\|x\|+\|y\| . \quad \forall x, y \in H
\end{gathered}
$$

Then the triangle inequality holds and complete the proof.

### 3.3 Relation between Hilbert space and Banach space

A Hilbert space is a Branch space with norm given by an inner (scalar) product

$$
\begin{equation*}
\|x\|=\sqrt{\langle x, x\rangle} \tag{*}
\end{equation*}
$$

For example, $\left(L^{2}(\mu),\|.\|_{2}\right)$ where
$\|x\|_{2}=\left(\int\|x\|^{2} d \mu\right)^{\frac{1}{2}}$.
So, we showed that Every Hilbert space is a Banach space.
But the question that should be asking now, Is it the converse true also?
In other words, is Every Banach space also Hilbert space?
The answer is that it is a necessary and sufficient for a Banach space to be Hilbert is for the norm stated above (*) should satisfy the parallelogram law.

### 3.4. Elementary Properties of Hilbert space

There are many inequalities and identities in Hilbert spaces those can provide some useful information to the operator theory in Hilbert spaces such as the Parallelogram identity,

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Polarization identity, Cauchy Schwarz inequality, The Pythagorean identity, etc. The parallelogram law plays a fundamental role in higher mathematics. As we just discussed in previous section the relation between Hilbert space and Banach, and we said that the essential condition for a Banach space to be Hilbert is achieving the parallelogram law.

### 3.4.1 Parallelogram Law

The definition of Parallelogram law can be defined as follows:
For any two elements $x$ and $y$ of an inner product space $H$ (Pre-Hilbert space) we have

$$
\begin{gathered}
\|x+y\|^{2}+\|x-y\|^{2}=2\left(\|x\|^{2}+\|y\|^{2}\right) \\
\text { Proof. }\|x+y\|^{2}+\|x-y\|^{2}= \\
<x+y, x+y>+<x-y, x-y> \\
=\langle x, x\rangle+\langle x, y>+<y, x\rangle+<y, y> \\
+<x, x>-<x, y>-<y, x\rangle+<y, y> \\
=2<x, x>+2<y, y> \\
\text { As }<x, x>=\|x\|^{2} \text { and }<y, y>=\|y\|^{2} \text { thus, } \\
\|x+y\|^{2}+\|x-y\|^{2}=2\|x\|^{2}+2\|y\|^{2} \\
=2\left(\|x\|^{2}+\|y\|^{2}\right)
\end{gathered}
$$

### 3.4.2 Polarization Identity

Let $x, y \in V$ be an innerproduct space. If $V$ is a vector space over $\mathbb{R}$ then

$$
4\langle x, y\rangle=\|x+y\|^{2}-\|x-y\|^{2}
$$

If $V$ is a vector space over $\mathbb{C}$ then

$$
\begin{gathered}
4<x, y>=\|x+y\|^{2}-\|x-y\|^{2} \\
-i\left(\|x+i y\|^{2}-\|x-i y\|^{2}\right)
\end{gathered}
$$

Proof. For a real vector space,

$$
\|x+y\|^{2}-\|x-y\|^{2}=
$$

$$
<x+y, x+y>-<x-y, x-y\rangle
$$

$$
=\langle x, x\rangle+\langle x, y\rangle+\langle y, x\rangle+\langle y, y\rangle
$$

$$
-(<x, x\rangle-<x, y>-<y, x\rangle+<y, y>)
$$

$$
=2\langle x, y\rangle+2<y, x\rangle
$$

$$
=2\langle x, y\rangle+2<x, y\rangle
$$

$$
=4\langle x, y\rangle
$$

For a complex vector space, note that by the same calculation as above.

$$
\|x+y\|^{2}-\|x-y\|^{2}=2\langle x, y\rangle+2\langle y, x\rangle
$$

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$$
\begin{aligned}
& \text { And, }\|x+i y\|^{2}-\|x-i y\|^{2}= \\
& 2\langle x, i y\rangle+2\langle i y, x\rangle \\
& \text { LHS }=2\langle x, y\rangle+2\langle y, x\rangle+ \\
& 2 i\langle x, i y\rangle+2 i\langle i y, x\rangle \\
& \text { LHS }=2\langle x, y\rangle+2\langle y, x\rangle+ \\
& 2 i\langle x, i y\rangle+2 i\langle i y, x\rangle \\
& =2\langle x, y\rangle+2\langle y, x\rangle+ \\
& 2(-i)^{2}\langle x, y\rangle+2 i^{2}\langle y, x\rangle \\
& 2\langle x, y\rangle+2\langle y, x\rangle+ \\
& 2\langle x, y\rangle-2\langle y, x\rangle \\
& =4\langle x, y\rangle
\end{aligned}
$$

3.4.3 Pythagorean Identity

Let $x, y \in H$, where His a pre-Hilbert space, then

$$
\begin{gathered}
<x, y>=0 \Rightarrow\|\mathrm{x}+\mathrm{y}\|^{2}=\|x\|^{2}+\|y\|^{2} \\
\text { Proof. }\|x+y\|^{2}=<x+\mathrm{y}, x+\mathrm{y}> \\
=<x, x>+<x, y>+<y, x>+<y, y> \\
\text { As }<x, y>=0 \text { so is }<y, x>\text {. Then } \\
\|x+y\|^{2}=<x, x>+<y, y> \\
=\|x\|^{2}+\|y\|^{2}
\end{gathered}
$$

### 3.4.4 Cauchy-Schwarz Inequality

Let Hbe a pre-Hilbert space. Then for all $x, y \in H$, we have

$$
\text { Assume } y \neq 0 \text {. Then }<y, y>\neq 0 \text {. }
$$

$$
\text { Let } \lambda=\frac{\langle x, y\rangle}{\langle y, y\rangle}, \forall \lambda \in \mathbb{C} .
$$

Then we have

$$
\begin{gathered}
0 \leq\|x-\lambda y\|^{2}=<x-\lambda y, x-\lambda y> \\
=<x, x>-\bar{\lambda}<x, y>-
\end{gathered}
$$

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$$
\begin{gathered}
\lambda<y, x\rangle+\lambda \bar{\lambda}<y, y> \\
=<x, x\rangle-\frac{\langle x, y><y, x\rangle}{\langle y, y\rangle} .
\end{gathered}
$$

Therefore, we have

$$
\begin{gathered}
|<x, y>|^{2}=\langle x, y\rangle\langle y, x\rangle \\
\Rightarrow 0 \leq<x, x\rangle-\frac{|<x, y>|^{2}}{\langle y, y>} \\
\frac{\|<x, y>\|^{2}}{\langle y, y>} \leq\langle x, x\rangle \\
\left.\Rightarrow|<x, y>|^{2} \leq<x, x\right\rangle<y, y> \\
=\|x\|^{2}\|y\|^{2}
\end{gathered}
$$

By taking the square root for both sides, we get

$$
\mathrm{\mid}\langle x, y>1 \leq\|x\|\|y\|
$$

### 3.4.5 Minkowski inequality

Let $H$ be a pre-Hilbert spaceand , $y \in H$. Then we have

$$
\|x+y\| \leq\|x\|+\|y\|
$$

Proof. $\|x+y\|^{2}=\langle x+y, x+y\rangle$
$=\langle x, x\rangle+\langle x, y\rangle+\langle y, x\rangle+\langle y, y\rangle$ (1)

$$
=\|x\|^{2}+\left\langle x, y>+\langle y, x\rangle+\|y\|^{2}\right.
$$

Now, from the Cauchy-Schwarz inequality just stated above.We have

$$
\mathrm{k} \mathrm{x}, \mathrm{y}>\mathrm{l} \leq\|x\|\|y\|
$$

Then we can say that

$$
<x, y>\leq\|x\|\|y\|
$$

Now we apply these results in our main equation (1). We get

$$
\begin{gathered}
\|x+y\|^{2} \leq\|x\|^{2}+2\|x\|\|y\|+\|y\|^{2}=(\|x\|+\|y\|)^{2} \\
\Rightarrow\|x+y\| \leq\|x\|+\|y\|
\end{gathered}
$$



### 3.4.6 Appolonius identity

Let $\mathrm{x}, \mathrm{y}, \mathrm{z}$ be three elements in a pre-Hilbert space called H.Then we have

$$
\begin{gathered}
\|z-x\|^{2}+\|z-y\|^{2}= \\
\frac{1}{2}\|x-y\|^{2}+\frac{1}{2}\|2 z-(x+y)\|^{2} \\
\text { Proof. }\|z-x\|^{2}+\|z-y\|^{2}= \\
<z-x, z-x>+<z-y, z-y> \\
=<z, z>-<z, x>-<x, z>+<x, x> \\
+<z, z>-<z, y>-<y, z>+<y, y> \\
=2\|z\|^{2}+\|x\|^{2}+\|y\|^{2}-<z, x> \\
-<x, z>-<z, y>-<y, z>\rightarrow(*) \\
\|x-y\|^{2}+\|2 z-(x+y)\|^{2}= \\
<x-y, x-y>+<2 z-(x+y), 2 z-(x+y)> \\
=<x, x>-<x, y>-<y, x>+<y, y> \\
+<2 z, 2 z>-<2 z, x+y> \\
-<x+y, 2 z>+<x+y, x+y> \\
=\|x\|^{2}+\|y\|^{2}+\|2 z\|^{2}-2<z, x+y> \\
-2<x+y, z>-<x, y>-<y, x> \\
+<x+y, x+y> \\
=\|x\|^{2}+\|y\|^{2}+\|2 z\|^{2}-2<z, x> \\
-2<z, y>-2<x, z>-2<y, z> \\
-<x, y>-<y, x>+<x, x> \\
+<x, y>+<y, x>+<y, y> \\
=\|x\|^{2}+\|y\|^{2}+\|2 z\|^{2}-2<z, x> \\
-2<z, y>-2<x, z> \\
<
\end{gathered}
$$

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$$
\begin{gathered}
-2<y, z>+\|x\|^{2}+\|y\|^{2} \\
=2\|x\|^{2}+2\|y\|^{2}+4\|z\|^{2}-2<z, x> \\
-2<z, y>-2<x, z>-2<y, z>
\end{gathered}
$$

therefore, we apply the results now

$$
\begin{gathered}
\frac{1}{2}\|x-y\|^{2}+\frac{1}{2}\|2 z-(x+y)\|^{2}= \\
\frac{1}{2}\left(\|x-y\|^{2}+\|2 z-(x+y)\|^{2}\right) \\
\left.=\|x\|^{2}+\|y\|^{2}+2\|z\|^{2}-<z, x\right\rangle \\
-<z, y>-<x, z>-<y, z>\rightarrow(* *)
\end{gathered}
$$

Now from (*),(**) we conclude that,

$$
\begin{gathered}
\|z-x\|^{2}+\|z-y\|^{2}= \\
\frac{1}{2}\|x-y\|^{2}+\frac{1}{2}\|2 z-(x+y)\|^{2}
\end{gathered}
$$

so, the prove is done and the inequality holds.
Definition 3.4.1. Let $V$ be an inner product space (pre- Hilbert space).
(i) Afamily $S$ of non-zero vectors in $V$ is called an orthogonal sequence if $x \perp y$ for any two distinct elements of $S \Leftrightarrow<x, y\rangle=0$.
(ii) A collection of vectors $\left(x_{\alpha}\right)_{\alpha \in A} \subseteq V$ is said to be orthogonal sequence if $\left\langle x_{\alpha}, x_{\beta}\right\rangle=0$ for all $\alpha \neq \beta$ and if $<x_{\alpha}, x_{\beta}>=1$ for all $\alpha=\beta$.

Theorem 3.4.1.Let $x_{1}, x_{2}, \ldots \ldots, x_{n}$ are orthogonal vectors in an inner productspace, then

$$
\left\|\sum_{k=1}^{n} x_{k}\right\|^{2}=\sum_{k=1}^{n}\left\|x_{k}\right\|^{2} .(*)
$$

Proof. $x_{1} \perp x_{2}$, then $\left\|x_{1}+x_{2}\right\|^{2}=\left\|x_{1}\right\|^{2}+\left\|x_{2}\right\|^{2}$ by identity (3.4.4). Thus, the theorem is true forn $=2$. Assume now that the (*)holds forn -1 , that is

$$
\left\|\sum_{k=1}^{n-1} x_{k}\right\|^{2}=\sum_{k=1}^{n-1}\left\|x_{k}\right\|^{2} .
$$

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$\operatorname{Set} x=\sum_{k=1}^{n-1} x_{k}$ and $y=x_{n^{*}}$

## Clearly $x \perp y$. Thus

$$
\begin{aligned}
& \left\|\sum_{k=1}^{n} x_{k}\right\|^{2}=\|x+y\|^{2}=\|x\|^{2}+\|y\|^{2} \\
& \quad=\sum_{k=1}^{n-1}\left\|x_{k}\right\|^{2}+\left\|x_{n}\right\|^{2}=\sum_{k=1}^{n}\left\|x_{k}\right\|^{2} .
\end{aligned}
$$

This theorem is called Pythagorean Formula.
Theorem 3.4.2 (Orthonormalization Process). Let $\left\{x_{n}\right\}$ be a sequence, finite or infinite, of linearly independent vectors in an inner product space $H$. Then there is an orthonormal sequence $\left\{e_{n}\right\}$ in $H$ such that for each $k$, both $\left\{x_{1}, x_{2}, \ldots \ldots, x_{k}\right\}$ and $\left\{e_{1}, e_{2}, \ldots \ldots . e_{k}\right\}$ generate the same vector subspace of $H$.

Proof.Let $G\left(z_{1}, z_{2}, \ldots z_{k}\right)$ denote the vector subspace spanned by the vectors $z_{1}, z_{2}, \ldots \ldots, z_{k} \in H$ and let $e_{1}=\frac{x_{1}}{\left\|x_{1}\right\|}$.Suppose $e_{1}, e_{2}, \ldots \ldots, e_{k}$ have been constructed by induction.

Let $a_{k+1}=x_{k+1}-\sum_{k=1}^{k}<x_{k+1}, e_{i}>e_{i}$. Suppose to the contrary that $a_{k+1}=0$. Then $x_{k+1} \in G\left(e_{1}, e_{2}, \ldots \ldots, e_{k}\right)=G\left(x_{1}, x_{2}, \ldots, x_{k}\right)$ gives the linear dependence of $x_{1}, x_{2}, \ldots, x_{k}, x_{k+1}$ which is a contradiction.

$$
\text { Hence, } a_{k+1} \neq 0 \text {. Define } e_{k+1}=\frac{a_{k+1}}{\left\|a_{k+1}\right\|^{*}} \text { clearly, }\left\|e_{k+1}\right\|=1
$$

$$
\begin{aligned}
& \text { Also for } 1 \leq p \leq k_{p} \text { we have }<\mathrm{e}_{\mathrm{k}+1}, e_{p}>= \\
& <x_{k+1}, e_{p}>-\sum_{i=1}^{k}<x_{k+1}, e_{j}><e_{j}, e_{p}> \\
& <x_{\mathrm{k}+1}, e_{p}>-\sum_{i=1}^{k}<x_{k+1}, e_{j}>\delta_{j p}= \\
& <x_{k+1}, e_{p}>-<x_{k+1}, e_{p}>=0
\end{aligned}
$$

Thus, $e_{1}, e_{2}, \ldots \ldots, e_{k}, e_{k+1}$ are orthogonal.
Now the following calculation completes the proof.

$$
G\left(e_{1}, e_{2}, \ldots, e_{k}, e_{k+1}\right)=G\left[G\left(e_{1}, e_{2}, \ldots, e_{k}\right), e_{k+1}\right]
$$

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$$
\begin{gathered}
=G\left[G\left(e_{1}, \ldots, e_{k}\right), x_{k+1}-\sum_{k=1}^{k}<x_{k+1}, e_{j}>e_{j}\right] \\
=G\left[G\left(e_{1}, e_{2}, \ldots, e_{k}\right), x_{k+1}\right] \\
=G\left[G\left(x_{1}, x_{2}, \ldots, x_{k}\right), x_{k+1}\right] \\
=G\left(x_{1}, x_{2}, \ldots, x_{k}, x_{k+1}\right)
\end{gathered}
$$

Lemma 3.4.1. let $\left\{x_{n}\right\}_{n=1}^{N}$ be an orthonormal system in an inner product space $V$ and let $\left\{a_{n}\right\}_{n=1}^{N}$ be a finite sequence of scalars. Then for all $x \in V$, we have

$$
\begin{gathered}
\left\|x-\sum_{n=1}^{N} a_{n} x_{n}\right\|^{2}=\|x\|^{2} \\
-\sum_{n=1}^{N}\left|<x, x_{n}>\left.\right|^{2}+\sum_{n=1}^{N}\right| a_{n}-<x, x_{n}>\left.\right|^{2}
\end{gathered}
$$

Proposition 3.4.1 (Bessel's Equality and Inequality).Let $\left(e_{i}\right)_{i \in N} \subset X$ be an orthonormal sequence in the pre-Hilbert space $X$. Then, for every $x \in X$ we have

$$
\begin{align*}
& \left\|x-\sum_{i=1}^{n}<x, e_{i}>e_{i}\right\|^{2}= \\
& \|x\|^{2}-\sum_{i=1}^{n}\left|<x, e_{i}>\right|^{2}  \tag{1}\\
& \sum_{i=1}^{n}\left|<x, e_{i}>\right|^{2} \leq\|x\|^{2} \tag{2}
\end{align*}
$$

Proof. In view of the Pythagorean identity (3.4.3), we have

$$
\begin{gathered}
\left\|\sum_{i=1}^{n}<x, e_{i}>e_{i}\right\|^{2}= \\
\sum_{i=1}^{n}\left\|<x, e_{i}>e_{i}\right\|^{2}=\sum_{i=1}^{n}\left|<x, e_{i}>\right|^{2} \\
\text { Hence, } \\
0 \leq\left\|x-\sum_{i=1}^{n}<x, e_{i}>e_{i}\right\|^{2}=
\end{gathered}
$$

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$$
\begin{gathered}
<x-\sum_{i=1}^{n}<x, e_{j}>e_{j}, x-\sum_{i=1}^{n}<x, e_{i}>e_{i}> \\
=<x, x>-<x, \sum_{i=1}^{n}<x, e_{i}>e_{i}> \\
-<\sum_{i=1}^{n}<x, e_{j}>e_{j}, x> \\
+<\sum_{I=1}^{n}<x, e_{j}>e_{j}, \sum_{i=1}^{n}<x, e_{i}>e_{i}> \\
=\|x\|^{2}-\sum_{i=1}^{n} \frac{<x, e_{i}><x, e_{i}>}{} \\
\quad-\sum_{i=1}^{\mathrm{n}}<x, e_{j}><x, e_{j}> \\
+\sum_{i=1}^{n} \sum_{i=1}^{n} \frac{<x, e_{j}><x, e_{i}><e_{j}, e_{i}>}{}
\end{gathered}
$$

Now, since is orthonormal. Then,

$$
0 \leq\|x\|^{2}-2 \sum_{i=1}^{n}\left|<x, e_{i}>\left.\right|^{2}+\sum_{i=1}^{n}\right|<x, e_{i}>\left.\right|^{2} \text { (3) }
$$

$$
=\|x\|^{2}-\sum_{i=1}^{n}\left|<x, e_{i}>\right|^{2}
$$

$$
\Rightarrow \text { equality (1) holds. Thus, from (3) }
$$

$$
\begin{aligned}
& \sum_{i=1}^{n}\left|<x, e_{i}>\right|^{2} \leq\|x\|^{2}= \\
& \sum_{i=1}^{\infty}\left|<x, e_{i}>\right|^{2} \quad \text { as } n \rightarrow \infty
\end{aligned}
$$

Which is complete the prove and the inequality (2) holds.
Theorem 3.4.3 (Riesz-Fischer). Let $\left(x_{n}\right)_{n \in N}$ be an ON-sequence (Orthonormal) in the Hilbert space $X$. Then,

$$
\sum_{n=1}^{\infty} c_{n} x_{n} \text { converges } \Leftrightarrow \sum_{n=1}^{\infty}\left|c_{n}\right|^{2}<\infty
$$

If this holds then we have

$$
\left\|\sum_{n=1}^{\infty} c_{n} x_{n}\right\|^{2}=\sum_{n=1}^{\infty}\left|c_{n}\right|^{2}
$$

First of all, we need to show that $\sum_{n=1}^{i} c_{n} x_{n}$ is a Cauchy sequence.

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Let's assume that $A_{i}=\sum_{n=1}^{i} c_{n} x_{n}$ for the partial sum.
Now, by Pythagoras we get $\forall j<i$ the following

$$
\begin{gathered}
\left\|A_{i}-A_{i}\right\|^{2}=<A_{i}-A_{i}, A_{i}-A_{j}> \\
=<\sum_{n=j+1}^{i} c_{n} x_{n}, \sum_{k=j+1}^{i} c_{k} x_{k}> \\
=\sum_{n=j+1}^{i} \sum_{k=j+1}^{i} c_{n} \bar{c}_{k}<x_{n}, c_{k}> \\
=\left\|\sum_{n=j+1}^{i} c_{n} x_{n}\right\|^{2} \\
=\sum_{n=j+1}^{i}\left|c_{n}\right|^{2} .
\end{gathered}
$$

This shows

$$
\begin{gathered}
\left\{A_{i}\right\} \text { is Cauchy } \Leftrightarrow \lim _{j, k \rightarrow \infty, j<k} \sum_{\mathrm{j}+1}^{k}\left|c_{n}\right|^{2}=0 \\
\Leftrightarrow \sum_{1}^{\infty}\left|c_{j}\right|^{2}<\infty .
\end{gathered}
$$

But as $X$ is complete,
$\Leftrightarrow\left\{A_{i}\right\}$ converges.
Now, by the continuity of the normas II. \||we mentioned early in proposition ... we get

$$
\begin{gathered}
\left\|\sum_{n=1}^{\infty} c_{n} x_{n}\right\|^{2}=\left\|\lim _{i} A_{i}\right\|^{2} \\
=\lim _{i}\left\|A_{i}\right\|^{2} \\
=\lim _{i} \sum_{n=1}^{i}\left|c_{n}\right|^{2}=\sum_{n=1}^{\infty}\left|c_{n}\right|^{2}
\end{gathered}
$$

Theorem 3.4.4. Let $\left\{x_{n}\right\}_{n=1}^{\infty}$ be an orthonormal system in a Hilbertspace $H$. Then the following are equivalent.
(1) $\left\{x_{n}\right\}$ is complete.
(2) for all $x \in H$,
$x=\sum_{n=1}^{\infty}<x, x_{n}>x_{n}$.
(3) $\|x\|^{2}=\sum_{n=1}^{\infty}\left|<x, x_{n}>\right|^{2}$

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## (Parseval identity)

(4) for all $x, y \in H$, we have,

$$
\left.<x, y\rangle=\sum_{n=1}^{\infty}<x, x_{n}>\ll y_{0} x_{n}\right\rangle
$$

Proof. Let $x \in H$. By Bessel's Equality (1) proposition 3.4.1. For every $n \in N$, we have

$$
\begin{aligned}
&\left\|x-\sum_{i=1}^{n}<x, e_{i}>e_{i}\right\|^{2} \\
&=\|x\|^{2}-\sum_{i=1}^{n}\left|<x, e_{i}>\right|^{2}(*) \\
& \operatorname{Let}\left\{x_{n}\right\}_{n}=\left\{e_{i}\right\}_{i} \\
& \Rightarrow\left\|x-\sum_{n=1}^{k}<x, x_{n}>x_{n}\right\|^{2} \\
&=\|x\|^{2}-\sum_{n=1}^{k}\left|<x_{i}, x_{n}>\right|^{2}
\end{aligned}
$$

If $\left\{x_{n}\right\}$ is complete, then the expression on the left in $\left({ }^{*}\right)$ converges to 0 as $k \rightarrow \infty$.
Hence,

$$
\lim _{k \rightarrow \infty}\left[\|x\|^{2}-\sum_{n=1}^{k}\left|<x, x_{n}>\right|^{2}\right]=0
$$

Therefore (3) (Parseval identity) holds.

## 4. Conclusion:

In this paper we have introduced the Hilbert spaces along with their properties, operations, and applications. Specifically, we have focused on significant and crucial spaces for Hilbert spaces called Banach Spaces. We have also studiedthe normed spaces and their properties. Finally, we discussed what is meant by Hilbert spaces and what is the relation between Hilbert and Banach spaces. In Future, we are planning to extend this work by exploring the orthogonal complement, projection and Rieszrepresentation Theorem.

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